

GEOMETRY OF WEBS OF ALGEBRAIC CURVES

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ABSTRACT. A family of algebraic curves covering a projective variety X is called a web of curves on X if it has only finitely many members through a general point of X . A web of curves on X induces a web-structure, in the sense of local differential geometry, in a neighborhood of a general point of X . We study how the local differential geometry of the web-structure affects the global algebraic geometry of X . Under two geometric assumptions on the web-structure, the pairwise non-integrability condition and the bracket-generating condition, we prove that the local differential geometry determines the global algebraic geometry of X , up to generically finite algebraic correspondences. The two geometric assumptions are satisfied, for example, when $X \subset \mathbb{P}^N$ is a Fano submanifold of Picard number 1, and the family of lines covering X becomes a web. In this special case, we have a stronger result that the local differential geometry of the web-structure determines X up to biregular equivalences. As an application, we show that if $X, X' \subset \mathbb{P}^N$, $\dim X' \geq 3$, are two such Fano manifolds of Picard number 1, then any surjective morphism $f : X \rightarrow X'$ is an isomorphism.

KEYWORDS. web geometry, extension of holomorphic maps, minimal rational curves, Fano varieties

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1. INTRODUCTION

Consider families of algebraic curves covering a projective variety X in such a way that there are only finitely many members of the family through a general point of X . We will call such a family a ‘web of curves’ (Definition 3.1) on X . In a Euclidean neighborhood of a general point of X , a web of curves on X induces a ‘web-structure’ (Definition 3.4), i.e. a finite collection of (1-dimensional) holomorphic foliations, a classical object in differential geometry. Although the study of web-structures has a long history in differential geometry (see [PP] and the references therein.), most of the existing theory is about web-structures of codimension 1. In this article, we will investigate *how the local differential geometry of the web-structure induced by a web of curves affects the global algebraic geometry* of the projective variety X . Our work suggests that the theory of 1-dimensional web-structures on manifolds of dimension ≥ 3 is a worthy subject of study.

The original motivation of this work was to prove the following.

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Theorem 1.1. *Let $X, X' \subset \mathbb{P}^N$ be two projective submanifolds of Picard number 1 covered by lines of \mathbb{P}^N . Let $\varphi : U \rightarrow U'$ be a biholomorphic map between two connected Euclidean open subsets $U \subset X$ and $U' \subset X'$ such that φ (resp. φ^{-1}) sends germs of lines in U (resp. U') to germs of lines in U' (resp. U). Then there exists a biholomorphic map (i.e. a biregular morphism) $\Phi : X \rightarrow X'$ such that $\varphi = \Phi|_U$.*

This was proved in [HM01] under the assumption that the family of lines passing through a general point of X and X' has positive dimension. The remaining part of Theorem 1.1, for which the method of [HM01] fails, is exactly when the families of lines on X and X' form webs of curves. This remaining part (and a more general version) has been raised as an open question in p. 566 of [HM01] and appeared as Question 5 in [Hw] in the list of major open problems in the study of minimal rational curves. Our Theorem 1.1 settles this remaining part. More explicitly, we can state the new component of Theorem 1.1 as follows.

Theorem 1.2. *Let $X, X' \subset \mathbb{P}^N$, $\dim X = \dim X'$, be two projective manifolds of Picard number 1 through a general point of which there are only finite, but nonzero, number of lines. Let \mathcal{W} (resp. \mathcal{W}') be a web of curves on X (resp. X') whose members are lines in \mathbb{P}^N . Let $\varphi : U \rightarrow U'$ be a biholomorphic map between two connected Euclidean open subsets $U \subset X$ and $U' \subset X'$ such that φ (resp. φ^{-1}) sends germs of lines belonging to \mathcal{W} (resp. \mathcal{W}') to germs of lines belonging to \mathcal{W}' (resp. \mathcal{W}). Then there exists a biholomorphic map (i.e. a biregular morphism) $\Phi : X \rightarrow X'$ such that $\varphi = \Phi|_U$.*

The condition that φ (resp. φ^{-1}) sends germs of lines in U (resp. U') to germs of lines in U' (resp. U) means that φ is an equivalence of the web-structures in the sense of local differential geometry. Thus Theorem 1.2 precisely says that, under the given assumptions, the local equivalence of the web-structures implies the biregular equivalence of the projective varieties. If we choose \mathcal{W} and \mathcal{W}' as the webs of all lines covering the projective manifolds, then Theorem 1.2 gives the remaining part of Theorem 1.1.

It is crucial that the open subsets U and U' in Theorem 1.2 are in Euclidean topology. As a matter of fact, if we replace Euclidean open subsets by Zariski open subsets in Theorem 1.2, the proof becomes straight-forward (see Proposition 8.7). Thus the key issue in the proof of Theorem 1.2 is to extend a holomorphic map defined on a Euclidean open subset to a Zariski open subset. We will achieve this in two steps:

- Step 1. Extension from a Euclidean open subset to an étale open subset.
- Step 2. Extension from an étale open subset to a Zariski open subset.

It turns out that our argument for Step 1 works in a much more general setting than Theorem 1.2 and proves the following.

Theorem 1.3. *Let X (resp. X') be a projective variety with a web \mathcal{W} (resp. \mathcal{W}') of curves. Assume that both \mathcal{W} and \mathcal{W}' are*

- (P) *pairwise non-integrable* (Definition 3.8) and
- (B) *bracket-generating* (Definition 4.2).

Let $\varphi : U \rightarrow U'$ be a biholomorphic map between two connected Euclidean open subsets $U \subset X$ and $U' \subset X'$ such that φ (resp. φ^{-1}) sends germs of members of \mathcal{W} in U (resp. \mathcal{W}' in U') to germs of members of \mathcal{W}' in U' (resp. \mathcal{W} in U). Then φ can be extended to a generically finite algebraic correspondence between X and X' , i.e., there exists a projective subvariety $\Gamma \subset X \times X'$ which contains the graph of φ and is generically finite over both X and X' .

This says that we can extend the complex analytic (or differential geometric) equivalence of the web-structures on Euclidean open subsets to an algebraic equivalence on étale open subsets, provided the webs satisfy two conditions (P) and (B). Both conditions are formulated as the failure of the involutiveness of certain distributions associated with the web-structures induced by the webs of curves. So these conditions are local differential geometric properties of the webs. But both of them can be interpreted also as algebro-geometric conditions on the webs of curves (Corollary 3.12 and Proposition 4.4, respectively).

The following examples show that both conditions (P) and (B) are necessary for Theorem 1.3. Fix two domains $O, O' \subset \mathbb{C}$ such that the restriction $f : O \rightarrow O'$ of the exponential map $e^z : \mathbb{C} \rightarrow \mathbb{C}$ is a biholomorphism.

Example 1.4. Set $X = X' = \mathbb{P}^1 \times \mathbb{P}^1$ and let \mathcal{W} be the web consisting of the two irreducible families of curves given by each factor of \mathbb{P}^1 . This web satisfies (B), but not (P). Set $U = O \times O$ and $U' = O' \times O'$. Then the product $(f, f) : U \rightarrow U'$ is a biholomorphic map preserving the web-structures, but cannot be extended to a generically finite correspondence.

Example 1.5. Use the terminology of Theorem 1.3. Consider projective varieties $Y := \mathbb{P}^1 \times X$ and $Y' := \mathbb{P}^1 \times X'$. They are equipped with webs \mathcal{V} and \mathcal{V}' induced by \mathcal{W} and \mathcal{W}' . Then \mathcal{V} and \mathcal{V}' satisfy (P), but not (B). Put $V = O \times U$ and $V' = O' \times U'$. The biholomorphic map $(f, \varphi) : V \rightarrow V'$ preserves the web-structures, but cannot be extended to a generically finite correspondence between Y and Y' .

The condition (B), in a different form, had appeared also in [HM01] and was used crucially in the extension argument there. Its role in the current work is very similar to that in [HM01], based on the construction (see Proposition 4.8) of a tower of auxiliary varieties by attaching members of the family of curves in an inductive way. A novel part of our argument in **Step 1** is to use the condition (P) to overcome the difficulty in applying the method of [HM01] in the current setting. Roughly speaking, the condition (P) provides the parameter space \mathcal{W} with a family of curves (see Corollary 3.12) whose members through a general point of \mathcal{W} form a positive-dimensional family. This situation is very similar to the main setting of [HM01], except that these curves are not necessarily determined by their tangent directions,

unlike the minimal rational curves considered in [HM01]. But this technical difference can be handled by using higher jets of curves (see Proposition 2.5) in place of their tangent directions and we can carry out the extension procedure in a way analogous to that of [HM01].

An important class of webs satisfying both (P) and (B) is étale webs of smooth rational curves (Definition 6.1) on Fano manifolds of Picard number 1. In particular, Theorem 1.3 implies the following general version of Theorem 1.2.

Theorem 1.6. *Fix two positive integers $\ell, \ell' > 0$. Let $X \subset \mathbb{P}^N$ (resp. $X' \subset \mathbb{P}^N$) be a projective manifold of Picard number 1 through a general point of which there are only finite, but nonzero, number of smooth rational curves of degree ℓ (resp. ℓ'). Let \mathcal{W} (resp. \mathcal{W}') be a web of curves on X (resp. X') general members of which are smooth rational curves of degree ℓ (resp. ℓ'). Let $\varphi : U \rightarrow U'$ be a biholomorphic map between two connected Euclidean open subsets $U \subset X$ and $U' \subset X'$ such that φ (resp. φ^{-1}) sends germs of rational curves belonging to \mathcal{W} (resp. \mathcal{W}') to germs of rational curves belonging to \mathcal{W}' (resp. \mathcal{W}). Then φ can be extended to a generically finite algebraic correspondence between X and X' , i.e., there exists an irreducible projective subvariety $\Gamma \subset X \times X'$ which contains the graph of φ and is generically finite over both X and X' .*

There are many Fano manifolds of Picard number 1 having such webs. In fact, all Fano threefolds of Picard number 1, excepting the 3-dimensional projective space and the 3-dimensional hyperquadric, have étale webs of smooth rational curves (see Chapter 4 of [IP]).

While Step 1 of the proof of Theorem 1.2 works in the general setting of Theorem 1.3, the argument in Step 2 for the extension from an étale open subset to a Zariski open subset is more subtle and does not work even in the setting of Theorem 1.6. To allow the argument in Step 2, the web should not be ‘pleated’ (see Definition 7.3), which is a global algebro-geometric condition. We verify this condition for Theorem 1.2 by exploiting a deformation-theoretic property of lines, which does not hold for rational curves of higher degree. In fact, the following example shows that we cannot expect Γ in Theorem 1.6 with $\ell, \ell' > 1$ to be the graph of a biregular morphism or even a rational map.

Example 1.7. Let $X_0 \subset \mathbb{P}^4$ be a smooth cubic threefold. There are exactly six lines through a general point of X_0 . Choose two general quadric hypersurfaces $Q, Q' \subset \mathbb{P}^4$ and let $f : X \rightarrow X_0$ (resp. $f' : X' \rightarrow X_0$) be the double cover of X_0 branched along $Q \cap X_0$ (resp. $Q' \cap X_0$). Then X, X' are Fano threefolds of Picard number 1 and the inverse images of lines on X_0 give rise to an étale web \mathcal{W} (resp. \mathcal{W}') of smooth rational curves on X (resp. X'). We can choose connected Euclidean open subsets $U_0 \subset X_0, U \subset X$ and $U' \subset X'$ such that $U_0 = f(U) = f'(U')$ and the restrictions

$$U \xrightarrow{f|_U} U_0 \xleftarrow{f'|_{U'}} U'$$

are biholomorphic. Then the composition

$$\varphi := (f'|_{U'})^{-1} \circ f|_U : U \rightarrow U'$$

is a biholomorphic map sending germs of members of \mathcal{W} to those of \mathcal{W}' . For general choices of Q and Q' , the two varieties X and X' cannot be biregular. The two morphisms f and f' give rise to a generically finite correspondence, predicted by Theorem 1.6, between X and X' .

The argument of Step 2 has the following application.

Theorem 1.8. *Fix a positive integer $\ell' > 0$. Let $X \subset \mathbb{P}^N$ (resp. $X' \subset \mathbb{P}^N$) be a projective manifold of Picard number 1 through a general point of which there are only finite, but nonzero, number of lines (resp. smooth rational curves of degree ℓ'). Assume that $\dim X = \dim X' \geq 3$. Then any surjective morphism $f : X \rightarrow X'$ is an isomorphism.*

Some special cases of Theorem 1.8 have been known. [Sc] proved it when $\dim X = \dim X' = 3$ under the additional assumption of the smoothness of the Hilbert scheme of lines. [HM03] proved it when $X = X'$, i.e., when f is a self-map. These special cases have been handled by arguments quite different from ours. To our knowledge, Theorem 1.8 is new even when X, X' are Fano complete intersections of index 2 of dimension ≥ 3 , which always satisfy the conditions of Theorem 1.8.

This paper is organized as follows. In Section 2, we present the basic definitions and some general results on covering families of curves. We introduce the condition (P) in Section 3 and the condition (B) in Section 4. The proof of Theorem 1.3 is given in Section 5. Section 6 is the verification that the webs in Theorem 1.6 satisfy the two conditions. In Section 7, we introduce the concept of pleated webs, which is the key in the argument of Step 2. Using this concept, we will prove Theorem 1.2 and Theorem 1.8 in Section 8.

Convention

1. We work over the complex numbers. Open sets and neighborhoods refer to Euclidean topology, unless otherwise specified.
2. An analytic (resp. algebraic) variety is an irreducible reduced complex space (resp. algebraic scheme). A Zariski open subset of an analytic variety means the complement of a closed analytic subset.
3. Let $f : X \rightarrow Y$ be a holomorphic map between varieties. For a nonsingular point $x \in X$ with $y = f(x)$ a nonsingular point of Y , we say that f is unramified (resp. submersive) at x if the derivative $df_x : T_x(X) \rightarrow T_y(Y)$ is injective (resp. surjective). When $f : X \dashrightarrow Y$ is a meromorphic map, we say that f is generically submersive if its germ at a general point of X is submersive, and f is generically biholomorphic if its germ at a general point of X is biholomorphic.

2. COVERING FAMILIES OF CURVES

Definition 2.1. Let M be a projective variety.

- (i) A projective subvariety $\mathcal{F} \subset \text{Chow}^1(M)$ of the Chow variety of 1-cycles on M is called an *irreducible covering family of curves* on M if the following conditions hold for the universal family morphisms $\rho_{\mathcal{F}} : \text{Univ}_{\mathcal{F}} \rightarrow \mathcal{F}$ and $\mu_{\mathcal{F}} : \text{Univ}_{\mathcal{F}} \rightarrow M$ (see I.3 of [Ko] for the definition of $\text{Univ}_{\mathcal{F}}$):
 - (1) a general fiber of $\rho_{\mathcal{F}}$ is irreducible and reduced; and
 - (2) $\mu_{\mathcal{F}}$ is surjective.
- (ii) A *covering family of curves* on M means a finite union of irreducible covering families of curves.
- (iii) We will denote by $\text{Univ}_{\mathcal{F}}^{\text{sm}}$ the dense Zariski open subset in $\text{Univ}_{\mathcal{F}}$ consisting of nonsingular points of $\text{Univ}_{\mathcal{F}}$ where the morphism $\rho_{\mathcal{F}}$ is smooth.

Definition 2.2. Let $\mathcal{F} \subset \text{Chow}^1(M)$ be a covering family of curves on a projective variety M .

- (i) For a surjective morphism $g : M \rightarrow M'$ to a projective variety M' that does not contract general members of any irreducible component of \mathcal{F} , the images under g of the general members of irreducible components of \mathcal{F} determine a covering family of curves on M' which will be denoted by $g_*\mathcal{F} \subset \text{Chow}^1(M')$. We have a natural dominant rational map $\text{univ}_{g_*\mathcal{F}} : \text{Univ}_{\mathcal{F}} \dashrightarrow \text{Univ}_{g_*\mathcal{F}}$.
- (ii) For a generically finite morphism $f : M' \rightarrow M$ from a projective variety M' , the inverse images under f of the general members of irreducible components of \mathcal{F} determine a covering family of curves on M' which will be denoted by $f^*\mathcal{F} \subset \text{Chow}^1(M')$. We have a natural generically finite rational map $\text{univ}_{f^*\mathcal{F}} : \text{Univ}_{f^*\mathcal{F}} \dashrightarrow \text{Univ}_{\mathcal{F}}$. We will denote by $\text{Bir}(f^*\mathcal{F})$ (resp. $\text{Mult}(f^*\mathcal{F})$) the union of irreducible components of $f^*\mathcal{F}$ general members of which are sent to members of \mathcal{F} birationally (resp. not birationally) by f , so that $f^*\mathcal{F} = \text{Bir}(f^*\mathcal{F}) \cup \text{Mult}(f^*\mathcal{F})$.

Note that in our definitions of $g_*\mathcal{F}$ and $f^*\mathcal{F}$, the images and the inverse images of members of \mathcal{F} are taken in the set-theoretical sense, not in cycle-theoretic sense.

Now we recall some facts on finite-order jet spaces of curves. The fiber bundle $\mathbb{J}^k(M)$ in the next definition is a Zariski open subset in the Semple k -jet bundle of $T(M)$ in the sense of [De]. We refer those who want a more precise presentation to Section 5 and Section 6 of [De], but we do not need the structure theory developed there.

Definition 2.3. Let M be a complex manifold. For a nonnegative integer $k \geq 0$, two germs of 1-dimensional submanifolds at a point $x \in M$ are *k -jet equivalent* if they have contact order at least k at x .

- (1) Denote by $\mathbb{J}_x^k(M)$ the complex manifold consisting of the k -jet equivalence classes of germs of 1-dimensional submanifolds at $x \in M$. This complex manifold $\mathbb{J}_x^k(M)$ has a natural structure of quasi-projective algebraic variety. The union $\mathbb{J}^k(M) = \cup_{x \in M} \mathbb{J}_x^k(M)$ is a holomorphic fiber bundle on M with a natural projection $\pi_M^k : \mathbb{J}^k(M) \rightarrow M$. For example, $\mathbb{J}^0(M) = M$ and $\mathbb{J}^1(M) = \mathbb{P}T(M)$.
- (2) A biholomorphic map between complex manifolds $f : M \rightarrow M'$ induces a biholomorphic fiber bundle morphism $d^k f : \mathbb{J}^k(M) \rightarrow \mathbb{J}^k(M')$ for each $k \geq 0$ satisfying the commuting diagram

$$\begin{array}{ccc} \mathbb{J}^k(M) & \xrightarrow{d^k f} & \mathbb{J}^k(M') \\ \pi_M^k \downarrow & & \downarrow \pi_{M'}^k \\ M & \xrightarrow{f} & M'. \end{array}$$

- (3) Let \overline{M} be an analytic variety and let M be its smooth locus. Then there exists an analytic variety $\mathbb{J}^k(\overline{M})$ with a meromorphic map $\pi_{\overline{M}}^k : \mathbb{J}^k(\overline{M}) \dashrightarrow \overline{M}$ such that the restriction of $\pi_{\overline{M}}^k$ to the inverse image of M is naturally isomorphic to π_M^k . A generically biholomorphic meromorphic map $f : \overline{M} \dashrightarrow \overline{M}'$ between analytic varieties induces a generically biholomorphic meromorphic map $d^k f : \mathbb{J}^k(\overline{M}) \dashrightarrow \mathbb{J}^k(\overline{M}')$ which agrees with $d^k f_x$ of (2) for the biholomorphic germ f_x of f at a general point $x \in M$.

Proposition 2.4. *Let \mathcal{F} be a covering family of curves on a projective variety M with the universal family $\rho_{\mathcal{F}} : \text{Univ}_{\mathcal{F}} \rightarrow \mathcal{F}$ and $\mu_{\mathcal{F}} : \text{Univ}_{\mathcal{F}} \rightarrow M$. Then for each $k \geq 0$, there exists a natural rational map $j_{\mathcal{F}}^k : \text{Univ}_{\mathcal{F}} \dashrightarrow \mathbb{J}^k(M)$ with the commuting diagram*

$$\begin{array}{ccc} \text{Univ}_{\mathcal{F}} & \xrightarrow{j_{\mathcal{F}}^k} & \mathbb{J}^k(M) \\ \mu_{\mathcal{F}} \downarrow & & \downarrow \pi_M^k \\ M & = & M. \end{array}$$

Furthermore, for sufficiently large k , the rational map $j_{\mathcal{F}}^k$ is generically injective on $\text{Univ}_{\mathcal{F}}$.

Proof. The map $j_{\mathcal{F}}^k$ is defined by considering the k -jet equivalence classes of the germs of members of \mathcal{F} at their nonsingular points. The commuting diagram is immediate from the definition. The generic injectivity of $j_{\mathcal{F}}^k$ for large k follows from the fact that for a fixed bounded family \mathcal{F} of curves on M , smooth germs of their members are determined by their k -jets for a sufficiently large k . \square

Proposition 2.5. *Let \mathcal{F} (resp. \mathcal{F}') be a covering family of curves on a projective variety M (resp. M') with the universal family $\rho_{\mathcal{F}} : \text{Univ}_{\mathcal{F}} \rightarrow \mathcal{F}$ (resp. $\rho_{\mathcal{F}'} : \text{Univ}_{\mathcal{F}'} \rightarrow \mathcal{F}'$) and $\mu_{\mathcal{F}} : \text{Univ}_{\mathcal{F}} \rightarrow M$ (resp. $\mu_{\mathcal{F}'} : \text{Univ}_{\mathcal{F}'} \rightarrow M'$). Let $\phi : V \dashrightarrow M'$ and $\psi : \mathbf{U} \dashrightarrow \text{Univ}_{\mathcal{F}'}$ be generically biholomorphic meromorphic maps from open subsets $V \subset M$ and $\mathbf{U} \subset \text{Univ}_{\mathcal{F}}$ such that*

- (1) \mathbf{U} intersects every irreducible component of $\text{Univ}_{\mathcal{F}}$;
- (2) $\mu_{\mathcal{F}}(\mathbf{U}) \subset V$;
- (3) the diagram

$$\begin{array}{ccccc} \text{Univ}_{\mathcal{F}} & \supset & \mathbf{U} & \xrightarrow{\psi} & \text{Univ}_{\mathcal{F}'} \\ \mu_{\mathcal{F}} \downarrow & & \downarrow & & \downarrow \mu_{\mathcal{F}'} \\ M & \supset & V & \xrightarrow{\phi} & M' \end{array}$$

commutes; and

- (4) $\phi(\mu_{\mathcal{F}}(\rho_{\mathcal{F}}^{-1}(\rho_{\mathcal{F}}(y)) \cap \mathbf{U})) \subset \mu_{\mathcal{F}'}(\rho_{\mathcal{F}'}^{-1}(\rho_{\mathcal{F}'}(\psi(y))))$ for a general $y \in \mathbf{U}$.

Then there exists a generically biholomorphic meromorphic map $\Psi : \mu_{\mathcal{F}}^{-1}(V) \dashrightarrow \text{Univ}_{\mathcal{F}'}$ such that

$$\begin{array}{ccccc} \text{Univ}_{\mathcal{F}} & \supset & \mu_{\mathcal{F}}^{-1}(V) & \xrightarrow{\Psi} & \text{Univ}_{\mathcal{F}'} \\ \mu_{\mathcal{F}} \downarrow & & \downarrow & & \downarrow \mu_{\mathcal{F}'} \\ M & \supset & V & \xrightarrow{\phi} & M' \end{array}$$

commutes and $\Psi|_{\mathbf{U}} = \psi$.

Proof. As in Definition 2.3 (3), the meromorphic map ϕ induces a meromorphic map

$$d^k \phi : \mathbb{J}^k(V) \dashrightarrow \mathbb{J}^k(M')$$

such that the induced map on the fiber

$$d_x^k \phi : \mathbb{J}_x^k(V) \longrightarrow \mathbb{J}_{\phi(x)}^k(M')$$

is biregular for a general $x \in V$. Let $j_{\mathcal{F}}^k : \text{Univ}_{\mathcal{F}} \dashrightarrow \mathbb{J}^k(M)$ (resp. $j_{\mathcal{F}'}^k : \text{Univ}_{\mathcal{F}'} \dashrightarrow \mathbb{J}^k(M')$) be the rational map defined in Proposition 2.4. Fix $k \gg 0$ such that both $j_{\mathcal{F}}^k$ and $j_{\mathcal{F}'}^k$ are generically injective. The conditions (2)-(4) on ψ imply that the following diagram of meromorphic maps commute:

$$\begin{array}{ccccccc} \mathbf{U} & \subset & \text{Univ}_{\mathcal{F}} & \xrightarrow{j_{\mathcal{F}}^k} & \mathbb{J}^k(M) & \supset & \mathbb{J}^k(V) \\ \psi \downarrow & & & & & & \downarrow d^k \phi \\ \text{Univ}_{\mathcal{F}'} & = & \text{Univ}_{\mathcal{F}'} & \xrightarrow{j_{\mathcal{F}'}^k} & \mathbb{J}^k(M') & = & \mathbb{J}^k(M'). \end{array}$$

Since $d^k \phi$ is biregular over a general point of V , it gives a generically biholomorphic meromorphic map between the proper images

$$j_{\mathcal{F}}^k(\mathbf{U}) \text{ and } j_{\mathcal{F}'}^k(\psi(\mathbf{U})).$$

By the condition (1), it gives a generically biholomorphic meromorphic map from the proper image $j_{\mathcal{F}}^k(\text{Univ}_{\mathcal{F}}) \cap (\pi_M^k)^{-1}(V)$ to $j_{\mathcal{F}'}^k(\text{Univ}_{\mathcal{F}'})$. As $j_{\mathcal{F}}^k$ and $j_{\mathcal{F}'}^k$ are generically injective, we have a meromorphic map $\Psi : \mu_{\mathcal{F}}^{-1}(V) \dashrightarrow \text{Univ}_{\mathcal{F}'}$ with the desired properties. \square

3. PAIRWISE NON-INTEGRABLE WEBS

Definition 3.1. Let M be a projective variety.

- (i) A covering family \mathcal{W} of curves on M is called a *web of curves* if the universal family morphism $\mu_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow M$ is generically finite. Thus a web of curves \mathcal{W} has pure dimension equal to $\dim M - 1$.
- (ii) Given two webs of curves \mathcal{W} and \mathcal{V} on M , we say that \mathcal{V} is a *subweb* of \mathcal{W} if $\mathcal{V} \subset \mathcal{W}$ as subsets of $\text{Chow}^1(M)$.
- (iii) A web \mathcal{W} of curves is *univalent* if $\mu_{\mathcal{W}}$ is birational.
- (iv) Any surjective morphism $f : M \rightarrow B$ to a projective variety B with $\dim M - \dim B = 1$ determines canonically an irreducible univalent web of curves on M , to be denoted by $\text{Fib}(f)$, a general member of which is an irreducible component of a general fiber of f .

Remark 3.2. Our definition of a web of curves is more general than the one used in [HM03]. The definition of a web given in [HM03] corresponds to an étale web, to be introduced in Section 6. Étale webs are much more restrictive, although our main applications, Theorem 1.2 and Theorem 1.6, are concerned with them.

The following is immediate by dimension-counting.

Lemma 3.3. *Let \mathcal{W} be a web of curves on a projective variety M with the universal family morphisms $\mu_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow M$ and $\rho_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow \mathcal{W}$. For any dense Zariski open subset $\mathcal{W}_o \subset \mathcal{W}$, there exists a dense Zariski open subset $M_o \subset M$ such that any member C of \mathcal{W} satisfying $C \cap M_o \neq \emptyset$ belongs to \mathcal{W}_o .*

The term ‘web’ originates from the notion of a web-structure in local differential geometry, defined as follows.

Definition 3.4. Let U be a complex manifold. A *web-structure* (of rank 1) on U is a finite collection of line subbundles

$$W_i \subset T(U), \quad 1 \leq i \leq d$$

for some integer $d \geq 1$ such that for any $1 \leq i \neq j \leq d$, the intersection $W_i \cap W_j \subset T(U)$ is the zero section. If we regard W_i as a 1-dimensional foliation on U , the condition implies that the leaves of W_i and W_j intersect transversally.

Proposition 3.5. *Let \mathcal{W} be a web of curves on a projective variety M with the universal family morphisms $\mu_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow M$ and $\rho_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow \mathcal{W}$. Let d be the degree of $\mu_{\mathcal{W}}$. Then there exists a Zariski open subset $M_{\text{reg}} \subset M$ such that each $x \in M_{\text{reg}}$ has an open neighborhood $\text{Reg}(x) \subset M_{\text{reg}}$ satisfying the following conditions.*

- (i) $\mu_{\mathcal{W}}^{-1}(\text{Reg}(x))$ consists of d disjoint connected open subsets

$$R_1, \dots, R_d \subset \text{Univ}_{\mathcal{W}}^{\text{sm}}$$

each of which is biholomorphic to $\text{Reg}(x)$ by $\mu_{\mathcal{W}}$.

- (ii) For each $1 \leq i \leq d$, each fiber of $\rho_{\mathcal{W}}|_{R_i}$ is connected.
- (iii) Let $W_i \subset T(\text{Reg}(x))$, $1 \leq i \leq d$, be the image of tangents to fibers of $\rho_{\mathcal{W}}|_{R_i}$. Then W_1, \dots, W_d give a web-structure on $\text{Reg}(x)$.

In particular, for any member C of \mathcal{W} , the intersection $C \cap \text{Reg}(x)$ is smooth.

Proof. The existence of a neighborhood $\text{Reg}(x)$ of a general point $x \in M$ satisfying (i) and (ii) is immediate from Definition 3.1. The line subbundles W_i defined in (iii) are distinct because $\mathcal{W} \subset \text{Chow}^1(M)$. Thus we can achieve (iii) by choosing M_{reg} suitably. \square

A fundamental notion in the local differential geometry is the equivalence of web-structures in the following sense.

Definition 3.6. Let U (resp. U') be a complex manifold with a web-structure $W_i \subset T(U)$, $1 \leq i \leq d$ (resp. $W'_j \subset T(U')$, $1 \leq j \leq d'$). A biholomorphic map $\varphi : U \rightarrow U'$ sends the web-structure W_i , $1 \leq i \leq d$ into the web-structure W'_j , $1 \leq j \leq d'$ if $d\varphi : T(U) \rightarrow T(U')$ sends each W_i to some W'_j . If furthermore $d = d'$, then we say that φ is an equivalence of the web-structures.

Definition 3.7. Let \mathcal{W} (resp. \mathcal{W}') be a web of curves on a projective variety M (resp. M'). Let $\varphi : U \rightarrow U'$ be a biholomorphic map between two open sets $U \subset M$ and $U' \subset M'$. For each point

$$x \in U \cap M_{\text{reg}} \cap \varphi^{-1}(M'_{\text{reg}}),$$

we can choose a neighborhood $U^x \subset U \cap \text{Reg}(x)$ of x (resp. $U'^{\varphi(x)} \subset U' \cap \text{Reg}(\varphi(x))$) with the web-structure induced by \mathcal{W} (resp. \mathcal{W}') as in Proposition 3.5. By shrinking U^x if necessary, we can assume that $\varphi(U^x) \subset U'^{\varphi(x)}$. We say that φ sends \mathcal{W} into \mathcal{W}' if the biholomorphic map $\varphi|_{U^x} : U^x \rightarrow U'^{\varphi(x)}$ sends the web-structure induced by \mathcal{W} into the web-structure induced by \mathcal{W}' for some (hence any by analyticity) x as above. We say that a meromorphic map $\psi : U \dashrightarrow U'$ sends \mathcal{W} into \mathcal{W}' if its biholomorphic germs at general points do so.

Definition 3.8. A web-structure $W_i \subset T(U)$, $1 \leq i \leq d$, on a complex manifold U is *pairwise non-integrable* if for each i , there exists $j \neq i$ such that the distribution $W_i + W_j \subset T(U)$ is not integrable. A web of curves \mathcal{W} on a projective variety X is *pairwise non-integrable* if the web-structure in $\text{Reg}(x)$ for each $x \in X_{\text{reg}}$ in Proposition 3.5 is pairwise non-integrable.

Remark 3.9. The term ‘pairwise non-integrable’ can be confusing, as it may suggest something different from Definition 3.8, for example, that $W_i + W_j$ is not integrable for each (or some) pair (i, j) , $i \neq j$. Since we do not have a good alternative term, we will use it by an abuse of language.

Definition 3.10. Let \mathcal{W} be a web of curves on a projective variety X with the universal family morphisms $\mu_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow X$ and $\rho_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow \mathcal{W}$. Fix an irreducible component \mathcal{V} of \mathcal{W} and write $f = \mu_{\mathcal{V}}$, $g = \rho_{\mathcal{V}}$ and $A =$

$\text{Univ}_{\mathcal{V}}$. The morphism $f : A \rightarrow X$ is generically finite and $g : A \rightarrow \mathcal{V}$ has connected 1-dimensional fibers. The inverse image $f^*\mathcal{W}$ is a web of curves on A .

- (i) Define a decomposition $f^*\mathcal{W} = \text{Fib}(g) \cup \text{Hor}(g)$ with no common components as follows. By Definition 3.1 (iv), the morphism g gives the univalent web $\text{Fib}(g)$ on A , which is a subweb of $f^*\mathcal{W}$. Write $\text{Hor}(g)$ for the union of components of $f^*\mathcal{W}$ different from $\text{Fib}(g)$, i.e., those components whose general members are ‘horizontal’ with respect to g .
- (ii) Because general members of components of $\text{Hor}(g)$ are not contracted by $g : A \rightarrow \mathcal{V}$, we can apply the construction in Definition 2.2 (i) to obtain a covering family of curves $g_*\text{Hor}(g)$ on \mathcal{V} and the induced rational map

$$\text{univ}_{g_*\text{Hor}(g)} : \text{Univ}_{\text{Hor}(g)} \dashrightarrow \text{Univ}_{g_*\text{Hor}(g)}.$$

- (iii) Define a decomposition $\text{Hor}(g) = \text{Inf}(g) \cup \text{Fin}(g)$ with no common components as follows. An irreducible component \mathcal{H} of $\text{Hor}(g)$ belongs to $\text{Fin}(g)$ (resp. $\text{Inf}(g)$) if the restriction of $\text{univ}_{g_*\text{Hor}(g)}$ in (ii) to $\text{Univ}_{\mathcal{H}}$ is generically finite (resp. not generically finite) over its image in $\text{Univ}_{g_*\text{Hor}(g)}$. In other words,

$$\text{Univ}_{\text{Hor}(g)} \xrightarrow{\text{univ}_{g_*\text{Hor}(g)}} \text{Univ}_{g_*\text{Hor}(g)}$$

is generically finite on $\text{Fin}(g)$ and has positive-dimensional fibers on $\text{Inf}(g)$.

Next proposition shows that members of $\text{Fin}(g)$ and $\text{Inf}(g)$ can be distinguished by a local property of the web-structure.

Proposition 3.11. *In the setting of Definition 3.10, let $A_{\text{reg}} \subset A$ be the Zariski open subset with respect to the web $f^*\mathcal{W}$ on A as in Proposition 3.5. For $y \in A_{\text{reg}}$, let $W_i, 1 \leq i \leq d$, be the web-structure on $\text{Reg}(y)$ induced by $f^*\mathcal{W}$. Assume that leaves of W_1 belong to the irreducible web $\text{Fib}(g)$ and the leaves of W_2 belong to an irreducible component \mathcal{H} of $\text{Hor}(g)$. Then $\mathcal{H} \subset \text{Inf}(g)$ if and only if the distribution $W_1 + W_2$ on $\text{Reg}(y)$ is integrable.*

Proof. We can find a member $C \subset A$ of \mathcal{H} through a general point $x \in \text{Reg}(y)$ such that $C \cap \text{Reg}(y)$ is the leaf of W_2 through x . Let F be the leaf of W_1 through x , which is an open subset in $g^{-1}(g(x))$.

If \mathcal{H} is a component of $\text{Inf}(g)$, then deformations of C intersecting F , say,

$$\{C_t, t \in \Delta, C_0 = C\}$$

are all sent by $g : A \rightarrow \mathcal{V}$ to the same curve in \mathcal{V} , i.e., $g(C_t) = g(C)$. Thus the germ of $g^{-1}(g(C))$ at x is the integral surface of the distribution $W_1 + W_2$.

Conversely, assume that $W_1 + W_2$ is integrable and let $S \subset \text{Reg}(y)$ be its 2-dimensional leaf through x . Then the germs of $g(S)$ and $g(C)$ at $g(x)$ coincide and the members of \mathcal{H} whose intersections with $\text{Reg}(y)$ are contained

in S must be sent to the curve $g(C)$ in \mathcal{V} . It follows that $\text{univ}_{g_*\text{Hor}(g)}$ is not generically finite on $\text{Univ}_{\mathcal{H}}$, that is, $\mathcal{H} \subset \text{Inf}(g)$. \square

Corollary 3.12. *A web of curves \mathcal{W} on a projective variety X is pairwise non-integrable if and only if $\text{Fin}(g)$ is not empty for any choice of \mathcal{V} in Definition 3.10.*

Proposition 3.13. *Let \mathcal{W} on X and \mathcal{W}' on X' be webs of curves on projective varieties. Fix an irreducible component \mathcal{V} of \mathcal{W} (resp. \mathcal{V}' of \mathcal{W}') and denote by $g : A \rightarrow \mathcal{V}$ and $f : A \rightarrow X$ (resp. $g' : A' \rightarrow \mathcal{V}'$ and $f' : A' \rightarrow X'$) the universal family morphisms as in Definition 3.10. We have the webs $\text{Fib}(g)$ and $\text{Fin}(g)$ on A (resp. $\text{Fib}(g')$ and $\text{Fin}(g')$ on A'). Let $\varphi : U \rightarrow U'$ be a biholomorphic map between connected open subsets $U \subset A$ and $U' \subset A'$, which sends $f^*\mathcal{W}$ into $(f')^*\mathcal{W}'$. If φ sends $\text{Fib}(g)$ into $\text{Fib}(g')$, then it sends $\text{Fin}(g)$ into $\text{Fin}(g')$.*

Proof. Since the problem is local, we may assume, by shrinking U and U' , that we have web-structures

$$W_1, \dots, W_d \subset T(U) \text{ and } W'_1, \dots, W'_d \subset T(U')$$

such that W_1 corresponds to $\text{Fib}(g)$ (resp. W'_1 corresponds to $\text{Fib}(g')$). By assumption, the differential $d\varphi$ sends W_1 to W'_1 and W_i to some W'_j . By Proposition 3.11, among W_i (resp. W'_i), $1 \leq i \leq d$, those corresponding to $\text{Fin}(g)$ (resp. $\text{Fin}(g')$) are characterized by the property that $W_1 + W_i$ (resp. $W'_1 + W'_i$) is not integrable. Since the integrability of such a distribution is preserved by $d\varphi$, we see that φ sends $\text{Fin}(g)$ into $\text{Fin}(g')$. \square

4. BRACKET-GENERATING WEBS

Definition 4.1. Let U be a complex manifold and let $D \subset T(U)$ be a distribution (Pfaffian system) on U , i.e., a vector subbundle of the tangent bundle of U . By the holomorphic Frobenius theorem (applied to the Pfaffian systems defined on Zariski open subsets of U generated by successive brackets of D), there exists a Zariski open subset $\text{dom}(D) \subset U$ and a holomorphic foliation, i.e., an integrable Pfaffian system,

$$\text{Fol}^D \subset T(\text{dom}(D)),$$

called the *foliation generated by D* , such that for a germ of complex submanifold $M \subset \text{dom}(D)$ if $D_x \subset T_x(M) \subset T_x(U)$ for each $x \in M$, then the germ of the leaf of Fol^D through a point $x \in M$ is contained in M . We say that D is *bracket-generating* if $\text{Fol}^D = T(\text{dom}(D))$. If, furthermore, U is a Zariski open subset in a projective variety X and D is an algebraic subbundle of $T(U)$, then $\text{dom}(D)$ is a Zariski open subset in X .

Definition 4.2. Given a web-structure W_1, \dots, W_d on a complex manifold U , the linear span $W_1 + \dots + W_d$ gives a vector subbundle $D \subset T(U)$ on

some Zariski open subset $U_o \subset U$. We say that the web-structure is *bracket-generating* if D is bracket-generating. Given a web \mathcal{W} of curves on a projective variety X , we have a Zariski open subset $X_o \subset X_{\text{reg}}$ and an algebraic vector subbundle $D^{\mathcal{W}} \subset T(X_o)$ spanned by the web-structures. Patching the data in Definition 4.1, we have a Zariski open subset $\text{dom}(D^{\mathcal{W}}) \subset X_o$ and a holomorphic foliation

$$\text{Fol}^{\mathcal{W}} := \text{Fol}^{D^{\mathcal{W}}} \subset T(\text{dom}(D^{\mathcal{W}}))$$

generated by $D^{\mathcal{W}}$. We say that \mathcal{W} is *bracket-generating* if the web-structure in $\text{Reg}(x)$ for each $x \in X_{\text{reg}}$ in Proposition 3.5 is bracket-generating. This is equivalent to saying that $\text{Fol}^{\mathcal{W}} = T(\text{dom}(D^{\mathcal{W}}))$.

Definition 4.3. Let X be a projective variety with a web \mathcal{W} of curves. For each $x \in X_{\text{reg}}$, let $I_x \subset X$ be the 1-dimensional closed algebraic subset defined by the union of all members of \mathcal{W} passing through x . We can choose a Zariski open subset $T \subset \text{dom}(D^{\mathcal{W}}) \subset X_{\text{reg}}$ such that for the incidence relation $I \subset T \times X$ defined by

$$I = \{(t, x), x \in I_t\},$$

the projection $\text{pr}_T : I \rightarrow T$ is flat. For a projective subvariety $Z \subset X$, define

$$I_Z := \text{closure of } \bigcup_{s \in Z \cap T} I_s.$$

Note that $I_Z = \emptyset$ if $Z \cap T = \emptyset$ and $I_Z = I_x$ when Z is one point $x \in T$. Although I_Z may not be irreducible, each component of I_Z contains Z by the flatness of $\text{pr}_T : I \rightarrow T$. Thus if $Z \cap T \neq \emptyset$, either $I_Z = Z$ or every component of I_Z has dimension equal to $\dim Z + 1$. We say that Z is *saturated* (with respect to \mathcal{W}) if $I_Z = Z$.

The following proposition shows that Definition 4.2 can be interpreted in terms of algebro-geometric property of the web.

Proposition 4.4. *In the setting of Definition 4.3, for each $x \in T$, there exists a minimal saturated subvariety $S_x \subset X$ through x , in the sense that any saturated subvariety through x contains S_x . Furthermore, the intersection $S_x \cap T$ is exactly the leaf of $\text{Fol}^{\mathcal{W}}|_T$ through x . It follows that \mathcal{W} is bracket-generating if and only if X itself is the only saturated subvariety passing through a general point of X .*

Proof. For each $x \in T$, we define a projective subvariety S_x containing x in the following manner. Let S_x^1 be a component of I_x and inductively define S_x^{i+1} to be a component of $I_{S_x^i}$. Recall from Definition 4.3 that for a projective variety $Z \subset X$, $Z \cap T \neq \emptyset$, either $I_Z = Z$ or every component of I_Z has dimension equal to $\dim Z + 1$. Thus $\dim S_x^{i+1} = \dim S_x^i + 1$ or $S_x^i = S_x^{i+1}$. We conclude that $S_x^n = S_x^{n+1} = \dots$, where $n = \dim X$. Define $S_x := S_x^n$. This is a saturated subvariety. Note that if $Z \subset X$ is saturated, then $I_{Z'} \subset Z$ for any subvariety $Z' \subset Z$. Thus if $x \in Z$, then $S_x^i \subset Z$ inductively for all

i and $S_x \subset Z$. This proves that S_x is the minimal saturated subvariety through x .

Let $S \subset T$ be a leaf of the foliation $\text{Fol}^{\mathcal{W}}|_T$, an immersed complex submanifold. If $y \in S$, then $I_y \cap T \subset S$ because each irreducible component of I_y is an integral curve of $D^{\mathcal{W}}$. Thus if a projective variety $Z \subset X$ satisfies $Z \cap T \subset S$, then $I_Z \cap T \subset S$. Denoting by $S(x)$ the leaf of $\text{Fol}^{\mathcal{W}}|_T$ through a point $x \in T$, we have $S_x^i \cap T \subset S(x)$ inductively for all i . This implies that $S_x \cap T \subset S(x)$ and

$$\dim S_x \leq \text{rank} \text{Fol}^{\mathcal{W}}.$$

On the other hand, if $Z \subset X$ is saturated, then for each nonsingular point $x \in Z \cap T$, we have $D_x^{\mathcal{W}} \subset T_x(Z)$. From Definition 4.1, we see that $S(x) \subset Z$. Applying this to the saturated variety $Z = S_y$ for $y \in T$, we have $S(x) \subset S_y$ for each nonsingular point $x \in S_y$. As $\dim S_y \leq \dim S(x)$, we conclude that $S_x \cap T$ is the leaf $S(x)$ of $\text{Fol}^{\mathcal{W}}|_T$ through x . \square

To exploit the bracket-generating property, we need the following geometric constructions.

Definition 4.5. Let \mathcal{V} be an irreducible web on a projective variety \mathbf{X} . To simplify the notation, denote by $A := \text{Univ}_{\mathcal{V}}$ the universal family and by $f : A \rightarrow \mathbf{X}$ (resp. $g : A \rightarrow \mathcal{V}$) the morphism $\mu_{\mathcal{V}}$ (resp. $\rho_{\mathcal{V}}$). Let \mathcal{E} be an irreducible web on A . Let $\eta : B \rightarrow A$ be a generically submersive holomorphic map from a normal analytic variety B . Then we can construct the following objects.

(1) Let

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & A \\ \varrho \downarrow & & \downarrow g \\ B & \xrightarrow{g \circ \eta} & \mathcal{V} \end{array}$$

be the normalization of the unique irreducible component of the pull-back $(g \circ \eta)^*A$ dominant over B . There is a canonical section $\sigma : B \rightarrow Z$ induced by η .

(2) Let

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\alpha}} & \text{Univ}_{\mathcal{E}} \\ \nu \downarrow & & \downarrow \mu_{\mathcal{E}} \\ Z & \xrightarrow{\alpha} & A \end{array}$$

be the normalization of an (not necessarily unique) irreducible component of the pull-back $\alpha^* \text{Univ}_{\mathcal{E}}$ dominant over Z .

We call

$$\begin{array}{ccccccc} Y & \xrightarrow{\nu} & Z & \xrightarrow{\varrho} & B \\ \tilde{\alpha} \downarrow & & \downarrow \alpha & & \downarrow g \circ \eta \\ \text{Univ}_{\mathcal{E}} & \xrightarrow{\mu_{\mathcal{E}}} & A & \xrightarrow{g} & \mathcal{V}. \end{array}$$

a $(\mathcal{V}, \mathcal{E})$ -construction on $\eta : B \rightarrow A$. It is not uniquely determined by $(\mathcal{V}, \mathcal{E})$, because the choice of ν in (2) is not unique.

We will apply this construction to $\eta : B \rightarrow A$ obtained in the following special way. Let \mathbf{Y} be an analytic variety and let $\chi : \mathbf{Y} \rightarrow \mathbf{X}$ be a generically submersive holomorphic map. Let

$$\begin{array}{ccc} B & \xrightarrow{\eta} & A \\ \zeta \downarrow & & \downarrow f \\ \mathbf{Y} & \xrightarrow{\chi} & \mathbf{X} \end{array}$$

be the normalization of an irreducible component (not necessarily unique) of the pull-back χ^*A dominant over \mathbf{Y} . A $(\mathcal{V}, \mathcal{E})$ -construction on $\eta : B \rightarrow A$ arising this way, with the notation $\tilde{\mathbf{X}} = \text{Univ}_{\mathcal{E}}$, $\tilde{\mathbf{Y}} = Y$, $\tilde{\chi} = \tilde{\alpha}$ and $h = \mu_{\mathcal{E}}$,

$$\begin{array}{ccccccc} \tilde{\mathbf{Y}} & \xrightarrow{\nu} & Z & \xrightarrow{e} & B & \xrightarrow{\zeta} & \mathbf{Y} \\ \tilde{\chi} \downarrow & & \downarrow \alpha & & \downarrow g \circ \eta & & \downarrow \chi \\ \tilde{\mathbf{X}} & \xrightarrow{h} & A & \xrightarrow{g} & \mathcal{V} & & \mathbf{X}, \end{array}$$

will be called a $(\mathcal{V}, \mathcal{E})$ -tower on $\chi : \mathbf{Y} \rightarrow \mathbf{X}$.

Definition 4.6. Let \mathcal{W} be a web on a projective variety X_0 of dimension n . Let $Y_0 \subset X_0$ be a connected open subset and denote by $\chi_0 : Y_0 \rightarrow X_0$ the inclusion. We will define inductively

- (i) a collection of projective varieties X_1, \dots, X_n with a generically finite surjective morphism $\lambda_i : X_i \rightarrow X_0$ for each $1 \leq i \leq n$;
- (ii) a collection of analytic varieties Y_1, \dots, Y_n with a projective morphism $\theta_i : Y_i \rightarrow Y_0$ of relative dimension i for each $1 \leq i \leq n$; and
- (iii) a generically submersive holomorphic map $\chi_i : Y_i \rightarrow X_i$ for each $1 \leq i \leq n$

in the following way.

Pick an irreducible component \mathcal{V}_1 of \mathcal{W} with the universal family $\mathcal{V}_1 \xleftarrow{g_1} A_1 \xrightarrow{f_1} X_0$ and an irreducible component \mathcal{E}_1 of $f_1^*\mathcal{W}$. Choose a $(\mathcal{V}_1, \mathcal{E}_1)$ -tower on $\chi_0 : Y_0 \rightarrow X_0$, to be denoted by

$$\begin{array}{ccccccc} Y_1 & \xrightarrow{\nu_1} & Z_1 & \xrightarrow{e_1} & B_1 & \xrightarrow{\zeta_1} & Y_0 \\ \chi_1 \downarrow & & \downarrow \alpha_1 & & \downarrow g_1 \circ \eta_1 & & \downarrow \chi_0 \\ X_1 & \xrightarrow{h_1} & A_1 & \xrightarrow{g_1} & \mathcal{V}_1 & & X_0. \end{array}$$

Denote by $\theta_1 : Y_1 \rightarrow Y_0$ the composition

$$Y_1 \xrightarrow{\nu_1} Z_1 \xrightarrow{e_1} B_1 \xrightarrow{\zeta_1} Y_0.$$

Then θ_1 is a projective morphism with relative dimension 1. Denote by $\lambda_1 : X_1 \rightarrow X_0$ the composition

$$X_1 \xrightarrow{h_1} A_1 \xrightarrow{f_1} X_0.$$

Then λ_1 is a generically finite morphism between two projective varieties.

To use an induction, assume that we have defined $\lambda_{i-1} : X_{i-1} \rightarrow X_0$, $\theta_{i-1} : Y_{i-1} \rightarrow Y_0$ and $\chi_{i-1} : Y_{i-1} \rightarrow X_{i-1}$ for $1 < i \leq n$. Pick an irreducible

component \mathcal{V}_i of $\lambda_{i-1}^* \mathcal{W}$ with the universal family $\mathcal{V}_i \xleftarrow{g_i} A_i \xrightarrow{f_i} X_{i-1}$ and an irreducible component \mathcal{E}_i of $(\lambda_{i-1} \circ f_i)^* \mathcal{W}$. Choose a $(\mathcal{V}_i, \mathcal{E}_i)$ -tower on $\chi_{i-1} : Y_{i-1} \rightarrow X_{i-1}$, to be denoted by

$$\begin{array}{ccccccc} Y_i & \xrightarrow{\nu_i} & Z_i & \xrightarrow{\varrho_i} & B_i & \xrightarrow{\zeta_i} & Y_{i-1} \\ \chi_i \downarrow & & \downarrow \alpha_i & & \downarrow g_i \circ \eta_i & & \downarrow \chi_{i-1} \\ X_i & \xrightarrow{h_i} & A_i & \xrightarrow{g_i} & \mathcal{V}_i & & X_{i-1}. \end{array}$$

Denote by $\theta_i : Y_i \rightarrow Y_0$ the composition

$$Y_i \xrightarrow{\nu_i} Z_i \xrightarrow{\varrho_i} B_i \xrightarrow{\zeta_i} Y_{i-1} \xrightarrow{\theta_{i-1}} Y_0.$$

Then θ_i is a projective morphism with relative dimension i . Denote by $\lambda_i : X_i \rightarrow X_0$ the composition

$$X_i \xrightarrow{h_i} A_i \xrightarrow{f_i} X_{i-1} \xrightarrow{\lambda_{i-1}} X_0.$$

Then λ_i is a generically finite morphism between two projective varieties.

The above inductive construction of objects (i), (ii), (iii) depends on the choice of $\mathcal{V}_i, \mathcal{E}_i$ and a choice of a $(\mathcal{V}_i, \mathcal{E}_i)$ -tower at each step. We will call this collection of objects a *tower* on $Y_0 \subset X_0$ constructed via $(\mathcal{V}_1, \dots, \mathcal{V}_n)$ and $(\mathcal{E}_1, \dots, \mathcal{E}_n)$.

A tower is equipped with the following special subsets $D_i \subset Y_i, 1 \leq i \leq n$, which may be viewed as the ‘diagonals’ of the construction.

Lemma 4.7. *For a tower on $Y_0 \subset X_0$ constructed in Definition 4.6, define inductively a closed analytic subset $D_i \subset Y_i$ for each $1 \leq i \leq n$ by setting $D_1 := \nu_1^{-1}(\sigma_1(B_1))$ and*

$$D_i := \nu_i^{-1}(\sigma_i(\zeta_i^{-1}(D_{i-1}))) \text{ for each } 1 < i \leq n,$$

where $\sigma_i : B_i \rightarrow Z_i$ is the natural section of ϱ_i induced by $\eta_i : B_i \rightarrow A_i$. Let $\xi_i : Y_i \rightarrow X_0$ be the composition $\lambda_i \circ \chi_i$. Then for each $1 \leq i \leq n$, we have

- (1) $\theta_i(D_i) = Y_0$ and
- (2) $\theta_i(y_i) = \xi_i(y_i)$ for any $y_i \in D_i$.

Proof. It is clear that $\theta_1(D_1) = Y_0$. Let us assume that $\theta_{i-1}(D_{i-1}) = Y_0$ for $1 < i \leq n$. Since

$$\varrho_i(\nu_i(D_i)) = \varrho_i(\sigma_i(\zeta_i^{-1}(D_{i-1}))) = \zeta_i^{-1}(D_{i-1}),$$

we have

$$\theta_i(D_i) = \theta_{i-1}(\zeta_i(\varrho_i(\nu_i(D_i)))) = \theta_{i-1}(D_{i-1}) = Y_0.$$

This proves (1).

To prove (2), note that for any $y_i \in D_i, 1 \leq i \leq n$, we have elements $b_i \in B_i$ and $y_{i-1} \in D_{i-1}$ (with the convention $D_0 = Y_0$) such that

$$\nu_i(y_i) = \sigma_i(b_i) \text{ and } \zeta_i(b_i) = y_{i-1}.$$

Then

$$\theta_1(y_1) = \zeta_1(\varrho_1(\nu_1(y_1))) = \zeta_1(b_1) = y_0,$$

while

$$\begin{aligned}
 \xi_1(y_1) &= \lambda_1(\chi_1(y_1)) \\
 &= (f_1 \circ h_1) \circ \chi_1(y_1) \\
 &= f_1(\alpha_1 \circ \nu_1(y_1)) \\
 &= f_1(\eta_1(b_1)) \\
 &= \chi_0(\zeta_1(b_1)) = y_0,
 \end{aligned}$$

checking (2) for $i = 1$. Now assume $\theta_{i-1}(y_{i-1}) = \xi_{i-1}(y_{i-1})$ for any $y_{i-1} \in D_{i-1}$. Then

$$\theta_i(y_i) = \theta_{i-1}(\zeta_i(\varrho_i(\nu_i(y_i)))) = \theta_{i-1}(\zeta_i(b_i)) = \theta_{i-1}(y_{i-1}),$$

while

$$\begin{aligned}
 \xi_i(y_i) &= \lambda_i \circ \chi_i(y_i) \\
 &= \lambda_{i-1} \circ (f_i \circ h_i) \circ \chi_i(y_i) \\
 &= \lambda_{i-1}(f_i(\alpha_i(\nu_i(y_i)))) \\
 &= \lambda_{i-1}(f_i(\eta_i(b_i))) \\
 &= \lambda_{i-1}(\chi_{i-1}(\zeta_i(b_i))) \\
 &= \lambda_{i-1}(\chi_{i-1}(y_{i-1})) = \xi_{i-1}(y_{i-1}).
 \end{aligned}$$

Thus (2) holds by induction. \square

The following is a key property of bracket-generating webs.

Proposition 4.8. *Let \mathcal{W} be a bracket-generating web on a projective variety X_0 of dimension n and let $Y_0 \subset X_0$ be a connected open subset. Then we can choose $(\mathcal{V}_1, \dots, \mathcal{V}_n)$ in Definition 4.6 such that for a tower on $Y_0 \subset X_0$ constructed via $(\mathcal{V}_1, \dots, \mathcal{V}_n)$ and $(\mathcal{E}_1, \dots, \mathcal{E}_n)$, under any choice of $(\mathcal{E}_1, \dots, \mathcal{E}_n)$, the morphism $\xi_i = \lambda_i \circ \chi_i : Y_i \rightarrow X_0$ sends each irreducible component of $\theta_i^{-1}(y)$ for a general $y \in Y_0$ to a projective subvariety of dimension i in X . In particular, the restriction of ξ_n on any irreducible component of $\theta_n^{-1}(y)$ is a generically finite morphism surjective over X_0 .*

Proof. We will choose $(\mathcal{V}_1, \dots, \mathcal{V}_n)$ inductively such that for each i and a general $y \in Y_0$,

$$\xi_i|_{\theta_i^{-1}(y)} : \theta_i^{-1}(y) \rightarrow X_0$$

sends each irreducible component of $\theta_i^{-1}(y)$ to a variety of dimension i . It is clear that we may choose \mathcal{V}_1 as any irreducible component of \mathcal{W} , to make this work for $i = 1$. Assuming that we have chosen $\mathcal{V}_1, \dots, \mathcal{V}_i, i < n$, satisfying the requirement, we will choose \mathcal{V}_{i+1} as follows.

Let $G_i \subset X_i$ be the image of an irreducible component of $\theta_i^{-1}(y)$ under χ_i . Our assumption is $\dim G_i = i < n$. Let $T_i \subset X_i$ be the Zariski open subset determined by the web $\lambda_i^* \mathcal{W}$ as in Definition 4.3 (by substituting X_i for X and $\lambda_i^* \mathcal{W}$ for \mathcal{W}). From the generic submersiveness of χ_i , the projective variety G_i intersects T_i . Since $\lambda_i^* \mathcal{W}$ is bracket-generating on X_i , Proposition

4.4 says that G_i is not saturated. Thus we can choose an irreducible component \mathcal{V}_{i+1} of $\lambda_i^* \mathcal{W}$ and an irreducible component J_{i+1} of $f_{i+1}^{-1}(G_i)$ which is dominant over G_i such that

$$\dim g_{i+1}(J_{i+1}) = \dim J_{i+1} = i$$

and, denoting by \tilde{J}_{i+1} the irreducible component of $g_{i+1}^{-1}(g_{i+1}(J_{i+1}))$ dominant over $g_{i+1}(J_{i+1})$,

$$\dim \tilde{J}_{i+1} = \dim f_{i+1}(\tilde{J}_{i+1}) = i + 1.$$

Choose any component \mathcal{E}_{i+1} of $(\lambda_i \circ f_{i+1})^* \mathcal{W}$. When $\chi_{i+1} : Y_{i+1} \rightarrow X_{i+1}$ is a $(\mathcal{V}_{i+1}, \mathcal{E}_{i+1})$ -tower on $\chi_i : Y_i \rightarrow X_i$, the image of B_{i+1} in A_{i+1} contains J_{i+1} . Then $\chi_{i+1}(\theta_{i+1}^{-1}(y))$ must have dimension $i + 1$ because it contains $f_{i+1}(\tilde{J}_{i+1})$, which has dimension $i + 1$. Thus some component of $\theta_{i+1}^{-1}(y)$ is sent by $\xi_{i+1} = \lambda_{i+1} \circ \chi_{i+1}$ to a variety of dimension $i + 1$. Since Y_i is irreducible and y is general, this holds for every component of $\theta_{i+1}^{-1}(y)$, completing the proof by induction. \square

5. PROOF OF THEOREM 1.3

For the proof of Theorem 1.3, it is convenient to introduce the following terms.

Definition 5.1. Let Q, R, S be three projective varieties of the same dimension.

- (1) A closed algebraic subset $\Gamma \subset S \times Q$ is called a *generically finite algebraic correspondence* from S to Q if the projections $\text{pr}_S : \Gamma \rightarrow S$ and $\text{pr}_Q : \Gamma \rightarrow Q$ are generically finite on every component of Γ . We will denote by $\mathcal{C}(S, Q)$ the set of generically finite algebraic correspondences from S to Q . For convenience, for an element $\gamma \in \mathcal{C}(S, Q)$, we will call the corresponding $\Gamma \subset S \times Q$ the *graph* of γ and write $\Gamma = \text{Graph}(\gamma)$. By symmetry, an element $\gamma \in \mathcal{C}(S, Q)$ can be viewed as an element of $\mathcal{C}(Q, S)$, to be denoted by γ^{-1} .
- (2) Given an element $\gamma \in \mathcal{C}(S, Q)$ and a general point $x \in S$, by composing pr_Q with the inverse images of $\text{pr}_S|_{\text{Graph}(\gamma)}$, we obtain a finite number of biholomorphic maps defined on a neighborhood O of x ,

$$\gamma_x^i : O \rightarrow \gamma_x^i(O) \subset Q, \quad 1 \leq i \leq m,$$

where $m = m(\gamma)$ is a positive integer depending on γ . The collection $\{\gamma_x^1, \dots, \gamma_x^m\}$ will be called the *germs* of γ at x .

- (3) Given $\gamma \in \mathcal{C}(R, S)$ and $\beta \in \mathcal{C}(S, Q)$, we denote by $\beta \circ \gamma \in \mathcal{C}(R, Q)$ the unique element canonically determined by the property that the germs of $\beta \circ \gamma$ at a general point $x \in R$ are the compositions of the germs $\{\gamma_x^1, \dots, \gamma_x^m\}$ of γ at x and the germs of β at the points $\gamma_x^1(x), \dots, \gamma_x^m(x)$.

- (4) Let U be an analytic variety. Let $\varphi : U \dashrightarrow S$ and $\psi : U \dashrightarrow Q$ be two generically submersive meromorphic maps. For an element $\gamma \in \mathcal{C}(S, Q)$, we write

$$\psi \sim \gamma \circ \varphi \text{ (or } \varphi \sim \gamma^{-1} \circ \psi),$$

if for a general point $y \in U$, the germ φ_y of φ at y and the germ ψ_y of ψ at y satisfy

$$\psi_y = \gamma_{\varphi(y)}^1 \circ \varphi_y$$

for some germ $\gamma_{\varphi(y)}^1$ of γ at $\varphi(y)$. In this case, for any generically submersive holomorphic map $h : V \rightarrow U$ from an analytic variety V , if we write $h^*\psi = \psi \circ h$ and $h^*\varphi = \varphi \circ h$, then it is clear that $h^*\psi \sim \gamma \circ h^*\varphi$, which may be written as

$$h^*(\gamma \circ \varphi) \sim \gamma \circ h^*\varphi.$$

If φ happens to be an inclusion of U as an open subset $\varphi : U \subset S$, then $\psi \sim \gamma \circ \varphi$ implies that $\text{Graph}(\psi) \subset U \times Q \subset S \times Q$ is contained in the closed algebraic subset $\text{Graph}(\gamma) \subset S \times Q$.

- (5) For an analytic variety U and meromorphic maps $\varphi_1 : U \dashrightarrow R$, $\varphi_2 : U \dashrightarrow S$ and $\varphi_3 : U \dashrightarrow Q$, if we have $\gamma \in \mathcal{C}(R, S)$ and $\beta \in \mathcal{C}(S, Q)$ such that $\varphi_2 \sim \gamma \circ \varphi_1$ and $\varphi_3 \sim \beta \circ \varphi_2$, then it is clear that

$$\varphi_3 \sim (\beta \circ \gamma) \circ \varphi_1.$$

- (6) Let \mathcal{W} be a web of curves on S . The *push-forward* of \mathcal{W} by $\gamma \in \mathcal{C}(S, Q)$, to be denoted by $\gamma_*\mathcal{W}$, is the web

$$\gamma_*\mathcal{W} := (\text{pr}_Q|_\Gamma)_*(\text{pr}_S|_\Gamma)^*\mathcal{W}$$

where $\Gamma = \text{Graph}(\gamma)$. Here, we are applying the pull-back and the push-forward to all irreducible components of Γ and taking the union of all the resulting webs. Note that \mathcal{W} is a subweb of, but not necessarily the same web as, the web $\gamma_*^{-1}(\gamma_*\mathcal{W})$.

Definition 5.2. Let $\alpha : Z \rightarrow A$ be a submersive holomorphic map between analytic varieties. Let $U \subset Z$ be an open subset and $\varphi : U \dashrightarrow Q$ be a meromorphic map to an analytic variety Q . We say that φ is α -*descending* if for each $u \in U$ there exists a germ $\varphi_{\alpha(u)}$ of meromorphic maps from A to Q at the point $\alpha(u) \in A$ such that $\varphi_{\alpha(u)} \circ \alpha$ coincides with the germ of φ at u . If furthermore, $\dim A = \dim Q$ and there are webs of curves \mathcal{W} on A and \mathcal{U} on Q , we say that φ *sends* \mathcal{W} into \mathcal{U} if $\varphi_{\alpha(u)}$ sends \mathcal{W} into \mathcal{U} at some (hence any) $u \in U$ in the sense of Definition 3.7.

Lemma 5.3. Let $\varrho : Z \rightarrow B$ be a proper holomorphic map between normal analytic varieties equipped with a section $\sigma : B \rightarrow Z$. Let $\alpha : Z \rightarrow A$ and $\eta : B \rightarrow A$ be generically submersive holomorphic maps to an analytic variety A , satisfying $\eta = \alpha \circ \sigma$. Let Q be an analytic variety and $\varphi : B \dashrightarrow Q$ be an η -descending meromorphic map. Then there exists a neighborhood $N(B) \subset Z$ of $\sigma(B)$ and an α -descending meromorphic map $\varphi^\sigma : N(B) \dashrightarrow Q$ such that $\varphi = \varphi^\sigma \circ \sigma$.

Proof. Since φ is η -descending, for each $u \in B$, we have a germ $\varphi_{\eta(u)}$ of meromorphic maps at $\eta(u)$ such that the germ of φ at u is just $\varphi_{\eta(u)} \circ \eta$. Let $(\varphi^\sigma)_{\sigma(u)}$ be the germ of meromorphic maps at $\sigma(u)$ given by $\varphi_{\eta(u)} \circ \alpha$. The collection of germs $(\varphi^\sigma)_{\sigma(u)}$ as u varies over B define the meromorphic map φ^σ in a neighborhood of $\sigma(B)$ with the desired properties. \square

Next proposition is a crucial step of the proof of Theorem 1.3.

Proposition 5.4. *Let \mathbf{X} be a projective variety with a web \mathcal{W} which is pairwise non-integrable. Let \mathcal{V} be an irreducible component of \mathcal{W} with the universal family morphisms $f : A \rightarrow \mathbf{X}$ and $g : A \rightarrow \mathcal{V}$, as in Definition 4.5. Fix a component \mathcal{E} of $\text{Fin}(g)$, which is nonempty because \mathcal{W} is pairwise non-integrable. Let B be a normal analytic variety and let $\eta : B \rightarrow A$ be a generically submersive holomorphic map. Let*

$$\begin{array}{ccccc} Y & \xrightarrow{\nu} & Z & \xrightarrow{\varrho} & B \\ \tilde{\alpha} \downarrow & & \downarrow \alpha & & \downarrow g \circ \eta \\ \text{Univ}_{\mathcal{E}} & \xrightarrow{\mu_{\mathcal{E}}} & A & \xrightarrow{g} & \mathcal{V}. \end{array}$$

be a $(\mathcal{V}, \mathcal{E})$ -construction on $\eta : B \rightarrow A$ from Definition 4.5. Let $N(B)$ be a neighborhood of $\sigma(B)$ where $\sigma : B \rightarrow Z$ is the canonical section of ϱ and let $\varphi : N(B) \dashrightarrow Q$ be an α -descending meromorphic map to a projective variety Q , which sends $f^*\mathcal{W}$ into a web \mathcal{U} on Q in the sense of Definition 5.2. Then there exists a projective variety S with $\gamma \in \mathcal{C}(S, Q)$ and an $\tilde{\alpha}$ -descending meromorphic map $\tilde{\varphi} : Y \dashrightarrow S$ such that for any $b \in B$ and any $x \in \nu^{-1}(\sigma(b))$, the germ $\tilde{\varphi}_x$ of $\tilde{\varphi}$ at x , the germ $\varphi_{\sigma(b)}$ of φ at $\sigma(b)$ and the germ ν_x of ν at x satisfy

$$(5.1) \quad \nu_x^* \varphi_{\sigma(b)} \sim \gamma \circ \tilde{\varphi}_x,$$

i.e., the following diagram, where we use a double arrow to denote the algebraic correspondence γ , commutes at the level of germs.

$$\begin{array}{ccccc} \text{Univ}_{\mathcal{E}} & \xleftarrow{\tilde{\alpha}} & Y & = & Y & \xrightarrow{\tilde{\varphi}} & S \\ \mu_{\mathcal{E}} \downarrow & & \downarrow \nu & & & & \Downarrow \gamma \\ A & \xleftarrow{\alpha} & Z & \supset & N(B) & \xrightarrow{\varphi} & Q. \end{array}$$

In particular, the $\tilde{\alpha}$ -descending map $\tilde{\varphi}$ sends the web $(f \circ \mu_{\mathcal{E}})^*\mathcal{W}$ into the web $(\gamma^{-1})_*\mathcal{U}$ on S in the sense of Definition 5.1 (6) and Definition 5.2.

Proof. By shrinking $N(B)$ if necessary, we may assume that each fiber of ϱ intersects $N(B)$ along a connected set. Since φ sends $f^*\mathcal{W}$ into \mathcal{U} , there exists an irreducible component, say \mathcal{V}' , of \mathcal{U} such that φ sends germs of fibers of ϱ at $\sigma(B)$ to germs of members of \mathcal{V}' , inducing a $(g \circ \eta)$ -descending meromorphic map $\varphi^\sharp : B \dashrightarrow \mathcal{V}'$. Setting $A' := \text{Univ}_{\mathcal{V}'}$, write $g' : A' \rightarrow \mathcal{V}'$ for $\rho_{\mathcal{V}'}$ and $f' : A' \rightarrow Q$ for $\mu_{\mathcal{V}'}$. Then φ induces an α -descending meromorphic map $\varphi' : N(B) \dashrightarrow A'$ such that $\varphi = f' \circ \varphi'$ and φ' sends $\text{Fib}(g)$ into $\text{Fib}(g')$.

It follows by Proposition 3.13 that φ' sends $\mathcal{E} \subset \text{Fin}(g)$ into $\text{Fin}(g')$.

$$\begin{array}{ccccc} & & N(B) & \xrightarrow{\varphi} & Q \\ & & \parallel & & \uparrow f' \\ A & \xleftarrow{\alpha} & Z & \supset & N(B) & \xrightarrow{\varphi'} & A' \\ g \downarrow & & \downarrow \varrho & & \downarrow & & \downarrow g' \\ \mathcal{V} & \xleftarrow{g \circ \eta} & B & = & B & \xrightarrow{\varphi^\sharp} & \mathcal{V}'. \end{array}$$

We have the covering families of curves $\mathcal{F} := g_*\mathcal{E}$ on \mathcal{V} and $\mathcal{F}' := (g')_*\text{Fin}(g')$ on \mathcal{V}' with the universal family morphisms

$$\begin{array}{ccccc} \text{Univ}_{\mathcal{E}} & \xrightarrow{\text{univ}_{g_*\mathcal{E}}} & \text{Univ}_{\mathcal{F}} & & \text{Univ}_{\text{Fin}(g')} & \xrightarrow{\text{univ}_{g'_*\text{Fin}(g')}} & \text{Univ}_{\mathcal{F}'} \\ \mu_{\mathcal{E}} \downarrow & & \downarrow \mu_{\mathcal{F}} & \text{and} & \mu_{\text{Fin}(g')} \downarrow & & \downarrow \mu_{\mathcal{F}'} \\ A & \xrightarrow{g} & \mathcal{V} & & A' & \xrightarrow{g'} & \mathcal{V}'. \end{array}$$

where the two rational maps on the first row are generically finite by the definition of $\text{Fin}(g)$ and $\text{Fin}(g')$.

Fix a point $b \in B$ and set $u = \sigma(b) \in \sigma(B)$. By the α -descending property of φ' , we have a neighborhood $\alpha(u) \in O \subset A$ with a meromorphic map $\varphi'_{\alpha(u)} : O \dashrightarrow A'$ such that $\varphi'_{\alpha(u)} \circ \alpha$ gives the germ of φ' at u . Similarly, we have a neighborhood $(g \circ \eta)(b) \in V \subset \mathcal{V}$ with a meromorphic map $\phi : V \dashrightarrow \mathcal{V}'$ such that $\phi \circ (g \circ \eta)$ gives the germ of φ^\sharp at b . We may assume that $g(O) \subset V$ by shrinking O if necessary. Let $\mathbf{U} \subset \text{Univ}_{\mathcal{E}}$ be a dense open subset in $\text{univ}_{g_*\mathcal{E}}(\mu_{\mathcal{E}}^{-1}(O))$. We have $\mu_{\mathcal{F}}(\mathbf{U}) \subset V$. Since φ' sends \mathcal{E} into $\text{Fin}(g')$, it induces a generically biholomorphic meromorphic map $\psi : \mathbf{U} \dashrightarrow \text{Univ}_{\mathcal{F}'}$ such that the diagram

$$\begin{array}{ccccc} \text{Univ}_{\mathcal{F}} & \supset & \mathbf{U} & \xrightarrow{\psi} & \text{Univ}_{\mathcal{F}'} \\ \mu_{\mathcal{F}} \downarrow & & \downarrow & & \downarrow \mu_{\mathcal{F}'} \\ \mathcal{V} & \supset & V & \xrightarrow{\phi} & \mathcal{V}' \end{array}$$

commutes and

$$\phi(\mu_{\mathcal{F}}(\rho_{\mathcal{F}}^{-1}(\rho_{\mathcal{F}}(y)) \cap \mathbf{U})) \subset \mu_{\mathcal{F}'}(\rho_{\mathcal{F}'}^{-1}(\rho_{\mathcal{F}'}(\psi(y))))$$

for a general $y \in \mathbf{U}$. Thus we can apply Proposition 2.5 to obtain a generically biholomorphic meromorphic map

$$\mu_{\mathcal{F}}^{-1}(V) \xrightarrow{\Psi} \mu_{\mathcal{F}'}^{-1}(V') \text{ satisfying } \psi = \Psi|_{\mathbf{U}}.$$

Then the composition

$$(g \circ \mu_{\mathcal{E}} \circ \tilde{\alpha})^{-1}(V) \xrightarrow{\tilde{\alpha}} (g \circ \mu_{\mathcal{E}})^{-1}(V) \xrightarrow{\text{univ}_{g_*\text{Fin}(g)}} \mu_{\mathcal{F}}^{-1}(V) \xrightarrow{\Psi} \text{Univ}_{\mathcal{F}'}$$

is a meromorphic map, to be denoted by $\tilde{\varphi}_{(\varrho \circ \nu)^{-1}(b)}$, defined in a neighborhood of $(\varrho \circ \nu)^{-1}(b)$ in Y . Since $\tilde{\varphi}_{(\varrho \circ \nu)^{-1}(b)}$ is uniquely determined by the germ of φ at u , the collection $\{\tilde{\varphi}_{(\varrho \circ \nu)^{-1}(b)}, b \in B\}$ defines a meromorphic

map $\tilde{\varphi} : Y \dashrightarrow \text{Univ}_{\mathcal{F}'}$. Let S be the irreducible component of $\text{Univ}_{\mathcal{F}'}$ where the image of $\tilde{\varphi}$ lies. Then there exists $\gamma \in \mathcal{C}(S, Q)$ given by the diagram

$$\begin{array}{ccccc}
 & & \text{Univ}_{\text{Fin}(g')} & \xrightarrow{\text{univ}_{g'_* \text{Fin}(g')}} & \text{Univ}_{\mathcal{F}'} \supset S \\
 & & \mu_{\text{Fin}(g')} \downarrow & & \downarrow \mu_{\mathcal{F}'} \\
 Q & \xleftarrow{f'} & A' & \xrightarrow{g'} & \mathcal{V}'
 \end{array}$$

where the first row, the first vertical arrow and the first horizontal arrow in the third row are generically finite. The property (5.1) is immediate from the construction. \square

We are ready to finish the proof of Theorem 1.3.

Proof of Theorem 1.3. The given conditions on $\varphi : U \rightarrow U'$ say that φ sends \mathcal{W} into \mathcal{W}' . Set $n = \dim X$, $Y_0 = U$ and $X_0 = X$. As the web \mathcal{W} is bracket-generating, we can find a tower on $Y_0 \subset X_0$ constructed via a certain choice of $(\mathcal{V}_1, \dots, \mathcal{V}_n)$ and any choice of $(\mathcal{E}_1, \dots, \mathcal{E}_n)$, satisfying the property stated in Proposition 4.8. Since \mathcal{W} is pairwise non-integrable, so is $(\lambda_{i-1} \circ f_i)^* \mathcal{W}$ on A_i . Thus we can assume that we have chosen \mathcal{E}_i as a component of $\text{Fin}(g_i) \subset (\lambda_{i-1} \circ f_i)^* \mathcal{W}$ for each i . Using the diagram from Definition 4.5 and Definition 4.6,

$$\begin{array}{ccccccc}
 & & & & A_i & \xrightarrow{f_i} & X_{i-1} \\
 & & & & \eta_i \uparrow & & \uparrow \chi_{i-1} \\
 Y_i & \xrightarrow{\nu_i} & Z_i & \xrightarrow{\varrho_i} & B_i & \xrightarrow{\zeta_i} & Y_{i-1} \\
 \chi_i \downarrow & & \downarrow \alpha_i & & \downarrow g_i \circ \eta_i & & \\
 X_i & \xrightarrow{h_i} & A_i & \xrightarrow{g_i} & \mathcal{V}_i & &
 \end{array}$$

and starting with β_0 defined as the identity $X'_0 := X'$, we will find inductively for each $1 \leq i \leq n$,

- (i) a projective variety X'_i and $\beta_i \in \mathcal{C}(X'_i, X')$;
- (ii) a generically submersive meromorphic map $\varphi_i : B_i \dashrightarrow X'_{i-1}$ which is η_i -descending and sends $(\lambda_{i-1} \circ f_i)^* \mathcal{W}$ into $(\beta_{i-1}^{-1})_* \mathcal{W}'$; and
- (iii) a generically submersive meromorphic map $\tilde{\varphi}_i : Y_i \dashrightarrow X'_i$ which is χ_i -descending and sends $\lambda_i^* \mathcal{W}$ into $(\beta_i^{-1})_* \mathcal{W}'$,

such that for any point $y_i \in D_i \subset Y_i$ of Lemma 4.7 with $y := \xi_i(y_i) \in Y_0$, the germ $\tilde{\varphi}_{i,y_i}$ of $\tilde{\varphi}_i$ at y_i , the germ φ_y of φ at y and the germ ξ_{i,y_i} of ξ_i at y_i satisfy

$$(5.2) \quad \xi_{i,y_i}^* \varphi_y \sim \beta_i \circ \tilde{\varphi}_{i,y_i},$$

or equivalently, the germ $\tilde{\varphi}_{i,x_i}$ at $x_i := \chi_i(y_i)$, representing $\tilde{\varphi}_{i,y_i}$ by the χ_i -descending property, satisfies

$$(5.3) \quad \lambda_{i,x_i}^* \varphi_y \sim \beta_i \circ \tilde{\varphi}_{i,x_i}$$

where λ_{i,x_i} is the germ of λ_i at x_i .

To start with, set

$$\varphi_1 := \zeta_1^* \varphi = (f_1 \circ \eta_1)^* \varphi : B_1 \dashrightarrow X'$$

which is η_1 -descending and sends $f_1^* \mathcal{W}$ into \mathcal{W}' . For a point $b_1 \in B_1$ with $\zeta_1(b_1) = y \in Y_0$, the germ $\varphi_{1, \eta_1(b_1)}$ at $\eta_1(b_1) \in A_1$ representing φ_1 satisfies

$$(5.4) \quad \varphi_{1, \eta_1(b_1)} = f_{1, \eta_1(b_1)}^* \varphi_y.$$

Applying Lemma 5.3, we have

$$\varphi_1^{\sigma_1} : N(B_1) \dashrightarrow X'$$

defined in a neighborhood $N(B_1)$ of $\sigma_1(B_1)$, which is α_1 -descending. By Proposition 5.4, we find $\gamma_1 \in \mathcal{C}(X'_1, X')$ and a χ_1 -descending meromorphic map

$$\tilde{\varphi}_1 = \widetilde{(\varphi_1^{\sigma_1})} : Y_1 \dashrightarrow X'_1$$

such that, by (5.1) of Proposition 5.4, for $y_1 \in \nu_1^{-1}(\sigma(b_1))$ and $x_1 = \chi_1(y_1)$,

$$h_{1, x_1}^* \varphi_{1, \eta_1(b_1)} \sim \gamma_1 \circ \tilde{\varphi}_{1, x_1}.$$

Then (5.4) and $\lambda_1 = f_1 \circ h_1$ give

$$\lambda_{1, x_1}^* \varphi_y \sim \gamma_1 \circ \tilde{\varphi}_{1, x_1}.$$

Thus (5.3) holds for $i = 1$ if we put $\beta_1 = \gamma_1$.

Assume that we have found $\beta_i \in \mathcal{C}(X'_i, X')$, φ_i and $\tilde{\varphi}_i$ satisfying (5.3) for $1 \leq i < n$. Define

$$\varphi_{i+1} = \zeta_{i+1}^* \tilde{\varphi}_i : B_{i+1} \dashrightarrow X'_i,$$

which is η_{i+1} -descending by $\chi_i \circ \zeta_{i+1} = f_{i+1} \circ \eta_{i+1}$ and sends $(\lambda_i \circ f_{i+1})^* \mathcal{W}$ into $(\beta_i^{-1})^* \mathcal{W}'$. This implies that for a point $y_i \in D_i$, $x_i := \chi_i(y_i)$ and $b_{i+1} \in B_{i+1}$ with $\zeta_{i+1}(b_{i+1}) = y_i$, the germ $\varphi_{i+1, \eta_{i+1}(b_{i+1})}$ at $\eta_{i+1}(b_{i+1}) \in A_{i+1}$ representing φ_{i+1} satisfies

$$(5.5) \quad \varphi_{i+1, \eta_{i+1}(b_{i+1})} = f_{i+1, \eta_{i+1}(b_{i+1})}^* \tilde{\varphi}_{x_i}.$$

Applying Lemma 5.3, we have

$$\varphi_{i+1}^{\sigma_{i+1}} : N(B_{i+1}) \dashrightarrow X'_i$$

defined in a neighborhood $N(B_{i+1})$ of $\sigma_{i+1}(B_{i+1})$, which is α_{i+1} -descending. By Proposition 5.4, we find $\gamma_{i+1} \in \mathcal{C}(X'_{i+1}, X'_i)$ and

$$\tilde{\varphi}_{i+1} = \widetilde{(\varphi_{i+1}^{\sigma_{i+1}})} : Y_{i+1} \dashrightarrow X'_{i+1}$$

which is χ_{i+1} -descending, such that for

$$y_{i+1} \in \nu_{i+1}^{-1}(\sigma_{i+1}(b_{i+1})) \in D_{i+1} \text{ and } x_{i+1} := \chi_{i+1}(y_{i+1}),$$

we have by (5.1),

$$(5.6) \quad h_{i+1, x_{i+1}}^* \varphi_{i+1, \eta_{i+1}(b_{i+1})} \sim \gamma_{i+1} \circ \tilde{\varphi}_{i+1, x_{i+1}}.$$

Then, in the notation of Definition 5.1 (4) and (5),

$$\begin{aligned}
\lambda_{i+1, x_{i+1}}^* \varphi_y &= (f_{i+1} \circ h_{i+1})_{x_{i+1}}^* (\lambda_{i, x_i}^* \varphi_y) \text{ from } \lambda_{i+1} = \lambda_i \circ f_{i+1} \circ h_{i+1} \\
&\sim \beta_i \circ \left((f_{i+1} \circ h_{i+1})_{x_{i+1}}^* \tilde{\varphi}_{i, x_i} \right) \text{ by the induction on (5.3)} \\
&\sim \beta_i \circ h_{i+1, x_{i+1}}^* \varphi_{i+1, \eta_{i+1}(b_{i+1})} \text{ by (5.5)} \\
&\sim (\beta_i \circ \gamma_{i+1}) \circ \tilde{\varphi}_{i+1, x_{i+1}} \text{ by (5.6)}.
\end{aligned}$$

Thus (5.3) holds for $i+1$ if we put $\beta_{i+1} = \beta_i \circ \gamma_{i+1}$. This completes the inductive construction of (i), (ii) and (iii) satisfying (5.2).

Now let F be a general fiber of $\theta_n : Y_n \rightarrow Y_0 = U$ such that the restriction $\tilde{\varphi}_n|_Y : F \dashrightarrow X'_n$ is a well-defined rational map, which we denote by Φ . The restriction $\xi_n|_F : F \rightarrow X$ is generically finite on each irreducible component of F by our choice of $(\mathcal{V}_1, \dots, \mathcal{V}_n)$ in Proposition 4.8. By (5.2) and $\xi_n(D_n) = Y_0$ from Lemma 4.7, there is an irreducible component R of F such that $\xi_n|_R$ regarded as an element of $\mathcal{C}(R, X)$ satisfies

$$\varphi \sim (\beta_n \circ \Phi|_R \circ (\xi_n|_R)^{-1}) \circ \iota$$

for the inclusion $\iota : U \subset X$. As mentioned in Definition 5.1 (4), this implies that the graph of φ is contained in the graph of a generically finite algebraic correspondence between X and X' . \square

6. ÉTALE WEBS OF SMOOTH CURVES

Definition 6.1. Let X be a projective manifold, i.e., a nonsingular projective variety. An *étale web of smooth curves* on X is a web \mathcal{W} of curves on X with the following additional property: in terms of the universal family $\mu_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow X$ and $\rho_{\mathcal{W}} : \text{Univ}_{\mathcal{W}} \rightarrow \mathcal{W}$, there exists a dense Zariski open subset $\mathcal{W}^{\text{étale}}$ of the smooth locus of \mathcal{W} such that for each $a \in \mathcal{W}^{\text{étale}}$,

- (i) $\rho_{\mathcal{W}}^{-1}(a)$ is a smooth curve;
- (ii) $\mu_{\mathcal{W}}|_{\rho_{\mathcal{W}}^{-1}(O_a)} : \rho_{\mathcal{W}}^{-1}(O_a) \rightarrow X$ is unramified for some open neighborhood O_a of a in $\mathcal{W}^{\text{étale}}$.

For a point $a \in \mathcal{W}^{\text{étale}}$, the smooth curve

$$P_a := \mu_{\mathcal{W}}(\rho_{\mathcal{W}}^{-1}(a)) \subset X$$

is called a *regular member of the web* \mathcal{W} . A regular member has trivial normal bundle by (ii). Conversely, it is easy to see that a web \mathcal{W} of curves on X is an étale web of smooth curves if a general member of \mathcal{W} is smooth and has trivial normal bundle in X . When we work with an étale web \mathcal{W} of smooth curves on X , we will choose X_{reg} of Proposition 3.5 such that $\mu_{\mathcal{W}}^{-1}(X_{\text{reg}}) \subset \rho_{\mathcal{W}}^{-1}(\mathcal{W}^{\text{étale}})$. This implies that any member of \mathcal{W} intersecting X_{reg} is a regular member. If regular members of the web are rational curves, we say that \mathcal{W} is an *étale web of smooth rational curves*.

Lemma 6.2. *Let \mathcal{W} be an irreducible étale web of smooth curves on a projective manifold X and let $H \subset X$ be an irreducible hypersurface which*

has positive intersection number with members of \mathcal{W} . Then there exist non-empty Zariski open subsets $H^{\mathcal{W}} \subset H$ and $\mathcal{W}^H \subset \mathcal{W}^{\text{étale}}$ such that

- (i) for any $a \in \mathcal{W}^H$, the regular member P_a intersects H transversally and $P_a \cap H \subset H^{\mathcal{W}}$;
- (ii) for any $x \in H^{\mathcal{W}}$, there exists $a \in \mathcal{W}^H$ with $x \in P_a$; and
- (iii) if a member $P \subset X$ of \mathcal{W} contains a point of $H^{\mathcal{W}}$, then $P = P_a$ for some $a \in \mathcal{W}^H$.

Proof. Choose a Zariski open subset $\mathcal{W}^o \subset \mathcal{W}^{\text{étale}}$ such that $\rho_{\mathcal{W}}^{-1}(\mathcal{W}^o) \cap \mu_{\mathcal{W}}^{-1}H$ is étale over \mathcal{W}^o . Set $H^o := \mu_{\mathcal{W}}(\rho_{\mathcal{W}}^{-1}(\mathcal{W}^o)) \cap H$. They satisfy the required conditions. \square

The following is immediate from Definition 6.1 (ii).

Lemma 6.3. *An étale web of smooth curves on a projective manifold of Picard number 1 cannot be univalent.*

The next lemma is proved in Proposition 6 of [HM03]. It follows from the fact that the base-change of an étale morphism is also an étale morphism.

Lemma 6.4. *Let $f : Y \rightarrow X$ be a generically finite morphism between two projective manifolds. If \mathcal{W} is an étale web of smooth curves on X , then $f^*\mathcal{W}$ is an étale web of smooth curves on Y .*

Proposition 6.5. *On a nonsingular projective surface, an irreducible étale web of smooth curves must be univalent.*

Proof. A regular member C of an étale web of smooth curves on a nonsingular projective surface satisfies $C \cdot C = 0$ because C has trivial normal bundle, as explained in Definition 6.1. If the web is not univalent, two distinct members C and C' through a general point satisfy $C \cdot C' > 0$. It follows that C and C' cannot belong to the same component of the web, i.e., the web cannot be irreducible. \square

Proposition 6.6. *Let \mathcal{W} be an étale web of smooth curves on a projective manifold X . Fix an irreducible component \mathcal{V} of \mathcal{W} and let $f : A = \text{Univ}_{\mathcal{V}} \rightarrow X$ and $g : A = \text{Univ}_{\mathcal{V}} \rightarrow \mathcal{V}$ be the universal family morphisms. The web $f^*\mathcal{W}$ on A has the subweb $\text{Fin}(g)$ from Definition 3.10 and the subweb $\text{Mult}(f^*\mathcal{W})$ of $f^*\mathcal{W}$ from Definition 2.2. Then $\text{Mult}(f^*\mathcal{W}) \subset \text{Fin}(g)$.*

Proof. Assuming that we have a common irreducible subweb \mathcal{J} of $\text{Inf}(g)$ and $\text{Mult}(f)$, we will derive a contradiction. Let $J \subset A$ be a general member of \mathcal{J} and let $W = g(J) \subset \mathcal{V}$. Since J is a member of $\text{Inf}(g)$, we have a family of distinct members $\{J_t, t \in \Delta, J = J_0\}$ of \mathcal{J} such that $W = g(J_t)$ for all $t \in \Delta$. Write $W^o = W \cap \mathcal{W}^{\text{étale}}$. Let $S \subset X$ be the closure of $f(g^{-1}(W^o))$. Then W^o determines a web of curves on S whose general members are given by the two morphisms $\rho_{W^o} : g^{-1}(W^o) \rightarrow W^o$ and $\mu_{W^o} : g^{-1}(W^o) \rightarrow S$ obtained by the restrictions of g and f . Choose a desingularization $\nu : \tilde{S} \rightarrow S$. Then the property (ii) of Definition 6.1 is preserved under the pull-back of μ_{W^o} by ν .

It follows that ρ_{W° and μ_{W° give rise to an irreducible étale web of smooth curves on \tilde{S} . By Proposition 6.5, this web must be univalent. This means that the morphism $\mu_{W^\circ} : g^{-1}(W^\circ) \rightarrow S$ is birational. But the closure of the surface $g^{-1}(W^\circ)$ is covered by the members $\{J_t, t \in \Delta\}$ of \mathcal{J} . Since the restriction of f to each J_t is not birational by the assumption $\mathcal{J} \subset \text{Mult}(f)$, the morphism μ_{W° cannot be birational, a contradiction. \square

Proposition 6.7. *Let X be a simply connected projective manifold of Picard number 1 equipped with an étale web \mathcal{W} of smooth curves. Let M be a projective variety and let $p : M \rightarrow X$ be a generically finite morphism which is not birational. Then $\text{Mult}(p^*\mathcal{W})$ is not empty.*

Proof. We may assume that M is nonsingular by taking desingularization. Let $H \subset X$ be an irreducible component of the reduced branch divisor of p , which is nonempty by the assumptions that X is simply connected and p is not birational. Let $R \subset M$ be an irreducible component of the ramification divisor of p such that $p(R) = H$. Since H is ample, there exists an irreducible component \mathcal{V} of $p^*\mathcal{W}$ whose members have positive intersection with R . Since \mathcal{V} is an étale web of smooth curves by Lemma 6.4, a general member C of \mathcal{V} intersects R transversally by Lemma 6.2 at a nonsingular point y of R . As $p(C)$ is a general member of $p_*\mathcal{V} \subset \mathcal{W}$, it intersects H transversally at a nonsingular point $p(y) \in H$, by Lemma 6.2 again. This implies that $p|_C : C \rightarrow p(C)$ is ramified at y , hence is not birational. It follows that $\mathcal{V} \subset \text{Mult}(p^*\mathcal{W})$. \square

Proposition 6.8. *Let \mathcal{W} be an étale web of smooth curves on a simply connected projective manifold X of Picard number 1. Fix an irreducible component \mathcal{V} of \mathcal{W} and let $f : \text{Univ}_{\mathcal{V}} \rightarrow X$ and $g : \text{Univ}_{\mathcal{V}} \rightarrow \mathcal{V}$ be the universal family morphisms. Then $\text{Mult}(f^*\mathcal{W}) \neq \emptyset$ and the web \mathcal{W} is pairwise non-integrable.*

Proof. Since X is nonsingular of Picard number 1, it cannot have a univalent étale web of smooth curves. It follows that the morphism $f : \text{Univ}_{\mathcal{V}} \rightarrow X$ is not birational. By Proposition 6.7, we see that $\text{Mult}(f^*\mathcal{W}) \neq \emptyset$. Then Proposition 6.6 shows $\text{Fin}(g) \neq \emptyset$, hence \mathcal{W} is pairwise non-integrable by Corollary 3.12. \square

The proof of the following proposition is essentially the same as that of Lemma 3.1 of [HM01].

Proposition 6.9. *Any étale web of smooth curves on a projective manifold of Picard number 1 is bracket-generating.*

Proof. Suppose that there exists a projective manifold X of Picard number 1 with an étale web \mathcal{W} of smooth curves which is not bracket-generating. By Proposition 4.4, there exists an irreducible subvariety \mathcal{H} of the Hilbert scheme of X whose general member is a saturated subvariety of X of dimension strictly smaller than X and whose members cover the whole X . Then by choosing a suitable subvariety of \mathcal{H} , we obtain a hypersurface $H \subset X$

which is the closure of the union of some collection of saturated subvarieties of X . Since X is of Picard number 1, members of each irreducible component \mathcal{V} of \mathcal{W} have positive intersection number with H . From Lemma 6.2, we have a Zariski open subset $H^\vee \subset H$, such that for any $b \in H^\vee$, we have a member P_a of \mathcal{V} intersecting H transversally at b . By our construction of H , we have a saturated subvariety $S \subset H$ with $S \cap H^\vee \neq \emptyset$. Pick a general point $b \in S$. Then we have P_a with $b \in P_a \cap S$ and $P_a \not\subset S$, which means that S is not saturated, a contradiction. \square

The following is well-known. We will give a proof for the reader's convenience.

Proposition 6.10. *Let $\ell > 0$ be a fixed integer. Let $X \subset \mathbb{P}^N$ be a projective submanifold such that there are nonempty, but only finitely many smooth rational curves of degree ℓ through a general point of X . Let \mathcal{W} be the web of curves on X such that members of \mathcal{W} through a general point $x \in X$ are exactly smooth rational curves of degree ℓ on X through x . Then \mathcal{W} is an étale web of smooth curves on X . If, furthermore, the Picard number of X is 1, then X is a Fano manifold and any étale web of smooth curves on X is a subweb of \mathcal{W} .*

Proof. It is well-known (see Chapter II of [Ko]) that the normal bundle of a smooth rational curve C of fixed degree ℓ through a general point x of the projective manifold X is semi-positive, i.e.,

$$N_{C \subset X} \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_{n-1}), \quad a_i \geq 0, \quad n = \dim X.$$

If $a_i > 0$ for some i , then denoting \mathbf{m}_x the maximal ideal at $x \in C$, we have

$$H^0(C, N_{C \subset X} \otimes \mathbf{m}_x) \neq 0 \text{ while } H^1(C, N_{C \subset X} \otimes \mathbf{m}_x) = 0.$$

By the basic deformation theory of submanifolds, this means that we can deform the rational curve C fixing the point x . This is a contradiction to the assumption that there are only finitely many smooth rational curves of degree ℓ through x . It follows that the normal bundle $N_{C \subset X}$ is trivial. This implies that \mathcal{W} is an étale web of smooth rational curves.

Now assume that the Picard group of X is generated by an ample line bundle L . The anti-canonical bundle K_X^{-1} of X is isomorphic to $L^{\mathbf{i}_X}$ for an integer \mathbf{i}_X (called the index of X) and the hyperplane line bundle of \mathbb{P}^N restricted to X is isomorphic to L^k for a positive integer k . Then for a general member C of \mathcal{W} ,

$$C \cdot L^k = \ell \text{ and } C \cdot L^{\mathbf{i}_X} = 2.$$

It follows that $\mathbf{i}_X = \frac{2k}{\ell}$ and the anti-canonical bundle is ample. For any étale web \mathcal{V} of smooth curves on X , a general member C' of \mathcal{V} has trivial normal bundle. Since $C' \cdot K_X^{-1} > 0$, we see that C' is a rational curve and $2 = C' \cdot L^{\mathbf{i}_X}$. This implies that $C' \cdot L^k = \ell$. Thus C' belongs to \mathcal{W} . \square

We are ready for the proof of Theorem 1.6.

Proof of Theorem 1.6. By Proposition 6.10, the manifold X and X' are Fano manifolds, thus they are simply connected. From Proposition 6.8 and Proposition 6.9, the webs \mathcal{W} and \mathcal{W}' satisfy the conditions (B) and (P) of Theorem 1.3, from which Theorem 1.6 follows. \square

7. PLEATED WEBS

Definition 7.1. Let \mathcal{W} be a web of curves on a projective variety X . Write $P_a := \mu_{\mathcal{W}}(\rho_{\mathcal{W}}^{-1}(a))$ for the curve in X corresponding to $a \in \mathcal{W}$. Let $P_a \neq P_b, a, b \in \mathcal{W}$, be two distinct members through a point $x \in X_{\text{reg}}$. Denote by $x(a)$ the unique intersection point $\mu_{\mathcal{W}}^{-1}(x) \cap \rho_{\mathcal{W}}^{-1}(a)$ and by $P_b^{x(a)}$ the unique irreducible component of $\mu_{\mathcal{W}}^{-1}(P_b)$ through $x(a)$. The image of the germ of the curve $x(a) \in P_b^{x(a)}$ under $\rho_{\mathcal{W}}$, which is a smooth germ of a 1-dimensional complex manifold through a in \mathcal{W} , will be denoted by $\text{Def}(P_a; P_b, x)$ and called the *deformation of P_a along P_b at x* . (The Zariski closure of $\text{Def}(P_a; P_b, x)$ is the curve $\rho_{\mathcal{W}}(P_b^{x(a)})$, which may have self-intersection at a .)

Lemma 7.2. *Let $f : Y \rightarrow X$ be a generically finite morphism between projective varieties. Let \mathcal{W} be a web of curves and let $f_b : f^*\mathcal{W} \dashrightarrow \mathcal{W}$ be the natural dominant rational map from the web $f^*\mathcal{W}$ on Y to \mathcal{W} . Let $Y_{\text{reg}} \subset Y$ be the Zariski open subset with respect to $f^*\mathcal{W}$ defined as in Proposition 3.5. Then there exists a dense Zariski open subset $Y_o \subset Y_{\text{reg}} \cap f^{-1}(X_{\text{reg}})$ such that for any $a, b \in f^*\mathcal{W}, a \neq b$, and $P_a \cap P_b \ni y \in Y_o$,*

$$f_b \text{Def}(P_a; P_b, y) = \text{Def}(f(P_a); f(P_b), f(y))$$

where the left hand side means the proper image under f_b .

Proof. From the definition of $f^*\mathcal{W}$ in Definition 2.2 (ii), there exists a dense Zariski open subset $\text{dom}(f_b) \subset f^*\mathcal{W}$ such that $f(P_a) = P_{f_b(a)}$ for any $a \in \text{dom}(f_b)$ and we have the commuting diagram of morphisms

$$\begin{array}{ccccc} \text{Univ}_{f^*\mathcal{W}} & \supset & \rho_{f^*\mathcal{W}}^{-1}(\text{dom}(f_b)) & \xrightarrow{\text{univ}_{f^*\mathcal{W}}} & \text{Univ}_{\mathcal{W}} \\ \rho_{f^*\mathcal{W}} \downarrow & & \downarrow & & \downarrow \rho_{\mathcal{W}} \\ f^*\mathcal{W} & \supset & \text{dom}(f_b) & \xrightarrow{f_b} & \mathcal{W}. \end{array}$$

Choose $Y_o \subset Y_{\text{reg}} \cap f^{-1}(X_{\text{reg}})$ such that $\mu_{f^*\mathcal{W}}^{-1}Y_o \subset \rho_{f^*\mathcal{W}}^{-1}(\text{dom}(f_b))$. If $P_a \cap P_b \ni y \in Y_o$ and $x = f(y)$, then $a, b \in \text{dom}(f_b)$ and it is easy to see that

$$\text{univ}_{f^*\mathcal{W}}(P_b^{y(a)}) = P_{f_b(b)}^{x(f_b(a))}.$$

This implies the desired result by the above commuting diagram. \square

Definition 7.3. In the setting of Definition 7.1, let $x \neq y$ be two distinct points of $P_a \cap X_{\text{reg}}$. We say that P_a is *pleated at $(x; y)$* if for any $b \in \mathcal{W}$ with $x \in P_b \neq P_a$, there exists $c \in \mathcal{W}$ ($b = c$ allowed) satisfying

$$y \in P_c \neq P_a \text{ and } \text{Def}(P_a; P_b, x) = \text{Def}(P_a; P_c, y).$$

An irreducible component \mathcal{V} of \mathcal{W} is a *pleated component* of \mathcal{W} if for a general member C of \mathcal{V} and a general point $x \in C$, there exists a point $y \in C \cap X_{\text{reg}}, x \neq y$, such that C is pleated at $(x; y)$. A web \mathcal{W} is *pleated* if it has a pleated component. A univalent web is pleated by definition.

Proposition 7.4. *Let \mathcal{W} be a pleated web on a projective variety X with a pleated component \mathcal{V} .*

- (1) *Suppose that for a member C of \mathcal{V} , we have an infinite subset $\mathcal{Z} \subset C \cap X_{\text{reg}}$ and a point $y \in C \cap X_{\text{reg}}$ such that C is pleated at $(z; y)$ for any $z \in \mathcal{Z}$. Then C is pleated at $(z; z')$ for infinitely many pairs $z \neq z'$ of elements of \mathcal{Z} .*
- (2) *Let $O \subset X$ be any dense Zariski open subset. Then a general member C of \mathcal{V} is pleated at $(x; y), x \neq y \in C \cap X_{\text{reg}}$ such that $x, y \in O$.*

Proof. For each $x \in C \cap X_{\text{reg}}$, denote by E_x^1, \dots, E_x^e all the members of \mathcal{W} through x different from C . For each $z \in \mathcal{Z}$, as C is pleated at $(z; y)$, we have (not necessarily distinct) integers $1 \leq z(1), \dots, z(e) \leq e$ such that

$$\text{Def}(C; E_z^i, z) = \text{Def}(C; E_y^{z(i)}, y) \text{ for each } 1 \leq i \leq e.$$

Since there are only finitely many choices of $z(1), \dots, z(e)$, we can assume that $z(i) = z'(i)$ for each $1 \leq i \leq e$ for infinitely many pairs $z \neq z'$ of elements of \mathcal{Z} . Then

$$\text{Def}(C; E_z^i, z) = \text{Def}(C; E_y^{z(i)}, y) = \text{Def}(C; E_y^{z'(i)}, y) = \text{Def}(C; E_{z'}^i, z')$$

for each $1 \leq i \leq e$. Thus C is pleated at $(z; z')$. This proves (i).

Given $O \subset X$, choose a member C of \mathcal{V} with $C \cap O \neq \emptyset$. Since C is pleated, we have an infinite subset $\mathcal{Z} \subset C \cap O$ and a point $y_z \in C \cap X_{\text{reg}}$ for each $z \in \mathcal{Z}$ such that C is pleated at $(z; y_z)$. If infinitely many of y_z 's are distinct, then $y_z \in O$ for some $z \in \mathcal{Z}$ and we are done. If infinitely many y_z 's coincide, we can deduce from (1) that C is pleated at $(z; z')$ for some pair $z, z' \in \mathcal{Z}$, proving (2). \square

Proposition 7.5. *Let $f : Y \rightarrow X$ be a generically finite morphism between projective varieties. Let \mathcal{W} be a web of curves on X . Then there exists a Zariski open subset $\mathcal{Y} \subset Y$ such that a member C of $\text{Mult}(f^*\mathcal{W})$ is pleated at $(x; y)$ if $x, y \in C \cap X_{\text{reg}} \cap \mathcal{Y}$ satisfy $x \neq y$ and $f(x) = f(y)$. It follows that any component of $\text{Mult}(f^*\mathcal{W})$ is a pleated component of $f^*\mathcal{W}$ and $f^*\mathcal{W}$ is pleated if $\text{Mult}(f^*\mathcal{W}) \neq \emptyset$.*

Proof. Let us use the terminology of Lemma 7.2. Using Lemma 3.3, choose a Zariski open subset $\mathcal{Y} \subset Y_o$ such that if $P_a \cap \mathcal{Y} \neq \emptyset$ for $a \in \mathcal{W}$, then $a \in \text{dom}(f_b)$ and f_b is unramified at a . Assuming that $x, y \in C \cap X_{\text{reg}} \cap \mathcal{Y}$ satisfy $x \neq y$ and $f(x) = f(y)$, let $E \neq C$ be any member of $f^*\mathcal{W}$ satisfying $x \in E \cap C$ and let F be an irreducible component (not necessarily different from E) of $f^{-1}(f(E))$ through y . By Lemma 7.2,

$$f_b \text{Def}(C; E, x) = \text{Def}(f(C); f(E), f(x)) =$$

$$\text{Def}(f(C); f(F), f(y)) = f_b \text{Def}(C; F, y).$$

As f_b is unramified at the point corresponding to C in $\text{dom}(f_b)$, this implies that $\text{Def}(C; E, x) = \text{Def}(C; F, y)$ as germs of curves in \mathcal{W} . It follows that C is pleated at $(x; y)$. \square

Proposition 7.6. *Let $f : Y \rightarrow X$ be a generically finite morphism between projective varieties and let $Y_o \subset Y$ be as in Lemma 7.2. Let \mathcal{W} be a web of curves on X such that $f^*\mathcal{W}$ is pleated with a pleated component \mathcal{V} . Assume that for a general member C of \mathcal{V} and a general point $z \in C \cap Y_o$, there is a point $z' \in C \cap Y_o$ such that*

$$f(z') \in X_{\text{reg}}, \quad z \neq z', \quad f(z) \neq f(z') \quad \text{and } C \text{ is pleated at } (z; z').$$

Then $f(C)$ is pleated at $(f(z); f(z'))$. It follows that \mathcal{W} is pleated, having $f_\mathcal{V}$ as a pleated component.*

Proof. Any member of \mathcal{W} through $f(z)$, different from $f(C)$, is of the form $f(E)$ for some member $E \neq C$ of $f^*\mathcal{W}$ through z . Since C is pleated at $(z; z')$, there exists a member F of $f^*\mathcal{W}$ satisfying $F \neq C, z' \in F$ and

$$\text{Def}(C; E, z) = \text{Def}(C; F, z').$$

By Lemma 7.2,

$$\text{Def}(f(C); f(E), f(z)) = f_b \text{Def}(C; E, z) =$$

$$f_b \text{Def}(C; F, z') = \text{Def}(f(C); f(F), f(z')).$$

By the assumption $f(z) \neq f(z') \in X_{\text{reg}}$, this implies that $f(C)$ is pleated at $(f(z); f(z'))$. \square

Proposition 7.7. *Let X be a simply connected projective manifold of Picard number 1 equipped with an étale web \mathcal{W} of smooth curves. Let X' be a projective variety equipped with a web \mathcal{W}' of curves. Let $\Gamma \subset X \times X'$ be a generically finite algebraic correspondence respecting $[\mathcal{W}; \mathcal{W}']$. If Γ is irreducible and the projection $\text{pr}_X : \Gamma \rightarrow X$ is not birational, then \mathcal{W}' is pleated.*

Proof. To simplify the notation, write $Y = \Gamma$ and the two projections as $f : Y \rightarrow X$ and $q : Y \rightarrow X'$. By assumption, the two webs $f^*\mathcal{W}$ and $q^*\mathcal{W}'$ on Y coincide. We know that $\text{Mult}(f^*\mathcal{W})$ is nonempty from Proposition 6.7. Let \mathcal{V} be any component of $\text{Mult}(f^*\mathcal{W})$. Let $\mathcal{Y} \subset Y$ be as in Proposition 7.5. For a general member C of \mathcal{V} and a general point $z \in C \cap \mathcal{Y}$, we can choose a point $z' \in \mathcal{Y}$ such that

$$f(z) = f(z'), \quad q(z) \neq q(z') \quad \text{and } q(z') \in X'_{\text{reg}}.$$

Then C is pleated at $(z; z')$ by Proposition 7.5. Thus $q_*\mathcal{V}$ is a pleated component of \mathcal{W}' by Proposition 7.6. \square

8. ÉTALE WEBS OF LINES

Definition 8.1. Let X be a projective manifold. An étale web \mathcal{W} of smooth rational curves is called an *étale web of lines* if there is an embedding $X \subset \mathbb{P}^N$ such that the image of the members of the web are lines in \mathbb{P}^N .

Proposition 8.2. Let \mathcal{W} be an étale web of lines on a projective manifold $X \subset \mathbb{P}^N$. A member $C \subset X$ of \mathcal{W} is said to be *free* if the normal bundle $N_{C \subset X}$ is trivial. Let $\widehat{\mathcal{W}}$ be the normalization of \mathcal{W} and let $\widehat{\mathcal{W}}^{\text{free}}$ be the dense Zariski open subset corresponding to free members. Let

$$\rho_{\widehat{\mathcal{W}}} : \text{Univ}_{\widehat{\mathcal{W}}} \rightarrow \widehat{\mathcal{W}} \text{ and } \mu_{\widehat{\mathcal{W}}} : \text{Univ}_{\widehat{\mathcal{W}}} \rightarrow X$$

be the normalization of $\rho_{\mathcal{W}}$ and $\mu_{\mathcal{W}}$. Then

- (i) $\rho_{\widehat{\mathcal{W}}}$ is a \mathbb{P}^1 -bundle;
- (ii) $\widehat{\mathcal{W}}^{\text{free}}$ is contained in the smooth locus of $\widehat{\mathcal{W}}$ and the morphism $\mu_{\widehat{\mathcal{W}}}$ is unramified on $\rho_{\widehat{\mathcal{W}}}^{-1}(\widehat{\mathcal{W}}^{\text{free}})$.

Proof. The normalization $\widehat{\mathcal{W}}$ corresponds to the union of finitely many components of the normalized space of rational curves $\text{RatCurves}^n(X)$ defined in II.2.11 of [Ko]. (i) is immediate because the members are lines (also from II.2.12 of [Ko]), while (ii) is from (i) and II.3.5.4 of [Ko]. \square

Proposition 8.3. In the setting of Proposition 8.2, write

$$P_a = \mu_{\widehat{\mathcal{W}}}(\rho_{\widehat{\mathcal{W}}}^{-1}(a)) \text{ for } a \in \widehat{\mathcal{W}}.$$

Fix an irreducible component \mathcal{V} of \mathcal{W} and let $f : \text{Univ}_{\widehat{\mathcal{V}}} \rightarrow X$ and $g : \text{Univ}_{\widehat{\mathcal{V}}} \rightarrow \widehat{\mathcal{V}}$ be the normalized universal family morphisms. Let $\mathcal{R} \subset \widehat{\mathcal{V}}$ be the union of all irreducible hypersurfaces $H \subset \widehat{\mathcal{V}}$ such that

- (1) $f(g^{-1}(H))$ is a hypersurface in X and
- (2) the morphism f is ramified at a general point of $g^{-1}(H)$.

Write $R = g^{-1}(\mathcal{R})$ and $B = f(R)$. Then there exist a dense Zariski open subset $B_o \subset B$ and a dense Zariski open subset $O \subset X$ with the following properties.

- (i) B_o is contained in the smooth locus of B and $g(f^{-1}(B_o) \cap R)$ is contained in the smooth locus of \mathcal{R} .
- (ii) If $a \in \mathcal{R}$ and $P_a \cap B_o \neq \emptyset$, then the normal bundle $N_{P_a \subset X}$ of $P_a \subset X$ is isomorphic to $\oplus_{1 \leq i \leq n-1} \mathcal{O}(m_i)$ for some integers m_i satisfying

$$m_1 \geq m_2 \geq \cdots \geq m_{n-2} \geq 0 > m_{n-1}.$$

- (iii) In (ii), for any point $x \in P_a \cap B_o$, the semipositive part of the fiber of the normal bundle of P_a at x

$$\oplus_{1 \leq i \leq n-2} \mathcal{O}(m_i)_x \subset \oplus_{1 \leq i \leq n-1} \mathcal{O}(m_i)_x \cong N_{P_a \subset X, x}$$

corresponds to

$$T_x(B_o)/T_x(P_a) \subset T_x(X)/T_x(P_a) = N_{P_a \subset X, x}.$$

(iv) In (iii), if $s \in H^0(P_a, N_{P_a \subset X})$, then

$$s_x \in T_x(B_o)/T_x(P_a) \subset T_x(X)/T_x(P_a) = N_{P_a \subset X, x}.$$

(v) Any member E of \mathcal{W} with $E \cap O \neq \emptyset$ is a regular member of \mathcal{W} , i.e., belonging to $\mathcal{W}^{\text{étale}}$, satisfies $E \cap B = E \cap B_o$ and intersects B_o transversally.

Proof. Denote by $\alpha : R \rightarrow B$ the restriction of f and by $\beta : R \rightarrow \mathcal{R}$ the restriction of g . Then α is a generically finite morphism and we can choose a dense Zariski open subset $B_o \subset B$ contained in the smooth locus of B such that

$$\alpha|_{\alpha^{-1}(B_o)} : \alpha^{-1}(B_o) \rightarrow B_o$$

is étale and the image $\beta(\alpha^{-1}(B_o))$ is contained in the smooth locus of \mathcal{R} . For $a \in \beta(\alpha^{-1}(B_o))$, write $N_{P_a \subset X} \cong \bigoplus_{i=1}^{n-1} \mathcal{O}(m_i)$ with $m_i \geq m_{i+1}$. Since a general member of \mathcal{W} has trivial normal bundle, we know that $K_X^{-1} \cdot P_a = 2$ and $m_1 + \dots + m_{n-1} = 0$. The sections of the trivial normal bundle of $\beta^{-1}(a)$ inside R give rise to elements of $H^0(P_a, N_{P_a \subset X})$ generating the subspace

$$T_x(B_o)/T_x(P_a) \subset N_{P_a \subset X, x}$$

at each point $x \in P_a \cap B_o$. It follows that $m_{n-2} \geq 0$. Then $m_{n-1} = -(m_1 + \dots + m_{n-2})$ is negative unless $m_1 = \dots = m_{n-1} = 0$. But the latter case cannot happen, because if $N_{P_a \subset X}$ is a trivial bundle, then f cannot be ramified at points of $g^{-1}(a)$ by Proposition 8.2 (ii), a contradiction to the choice $a \in \mathcal{R}$. Thus $m_{n-1} < 0$ and $\bigoplus_{1 \leq i \leq n-2} \mathcal{O}(m_i)_x$ should correspond to $T_x(B_o)/T_x(P_a)$. This verifies (i)-(iv). (iv) is clear from Lemma 6.2. \square

Proposition 8.4. *In the setting of Proposition 8.3, assume furthermore that X has Picard number 1. Then \mathcal{R}, R, B and $\text{Mult}(f^*\mathcal{W})$ are nonempty. A general member of any component of $\text{Mult}(f^*\mathcal{W})$ intersect R transversally at points of $f^{-1}(B_o)$.*

Proof. Since X has Picard number 1 and is covered by lines, it must be Fano, which implies that X is simply connected. Thus the sets \mathcal{R}, R and B in Proposition 8.3 are not empty and Proposition 6.8 shows that $\text{Mult}(f^*\mathcal{W})$ is not empty. For a general member E of $\text{Mult}(f^*\mathcal{W})$, the morphism $f|_E : E \rightarrow f(E)$ is a branched cover of a line. Thus E must intersect the ramification locus of f . Since \mathcal{W} is an étale web of lines, the images of ramification locus of $f|_E$ must cover a hypersurface in X as E varies in $\text{Mult}(f^*\mathcal{W})$. Thus E has nonempty intersection with R . The intersection is transversal, because $f(E)$ should intersect B transversally by Lemma 6.2. \square

Theorem 8.5. *Let $X \subset \mathbb{P}^N$ be a projective submanifold of Picard number 1 and let \mathcal{W} be an étale web of lines on X . Then \mathcal{W} is not pleated.*

Proof. Let us assume that \mathcal{W} has a pleated component \mathcal{V} and derive a contradiction. We will use the notation of Proposition 8.3 and Proposition 8.4.

For a general member E of $\text{Mult}(f^*\mathcal{W})$, intersecting R transversally at points of $f^{-1}(B_o)$, pick a point $z \in E$ with $f(z) \in O$ and let $C \subset X$ be the member of \mathcal{V} corresponding to $g(z)$. Since C is pleated, there exists $y \in g^{-1}(g(z))$ such that C is pleated at $(f(z); f(y))$. Moreover, we can assume that $f(y) \in O$ by Proposition 7.4. Thus we have a member $F \subset \text{Univ}_{\widehat{\mathcal{V}}}$ of $f^*\mathcal{W}$ through y such that

$$\text{Def}(C; f(E), f(z)) = \text{Def}(C; f(F), f(y)).$$

This implies that $g(E) = g(F)$ in $\widehat{\mathcal{V}}$. This curve $g(E)$ intersects the hypersurface $\mathcal{R} \subset \widehat{\mathcal{V}}$ transversally at points in $g(f^{-1}(B_o) \cap R)$.

Since $f(z)$ and $f(y)$ are two distinct points on the line C , the line $f(E)$ through $f(z)$ and the line $f(F)$ through $f(y)$ must be different. Suppose that $f(E) \cap f(F) \neq \emptyset$. Then the family of lines on X parametrized by $g(E)$ lie on the plane $\langle f(E), f(F) \rangle \subset \mathbb{P}^N$ spanned by $f(E)$ and $f(F)$. Thus $\langle f(E), f(F) \rangle \subset X$. But then the line C on this plane cannot have trivial normal bundle in X , a contradiction. Thus we have $f(E) \cap f(F) = \emptyset$.

Let $\iota : \Delta = \{t \in \mathbb{C}, |t| < 1\} \rightarrow g(E) \subset \widehat{\mathcal{V}}$ be a local uniformization of the curve $g(E)$ at the point $\iota(0) \in g(E) \cap \mathcal{R}$. The pull-back of the \mathbb{P}^1 -bundle $g : \text{Univ}_{\widehat{\mathcal{V}}} \rightarrow \widehat{\mathcal{V}}$ by ι is biholomorphic to a trivial bundle $p : \mathbb{P}^1 \times \Delta \rightarrow \Delta$, equipped with a natural holomorphic map $j : \mathbb{P}^1 \times \Delta \rightarrow \text{Univ}_{\widehat{\mathcal{V}}}$:

$$\begin{array}{ccccc} \mathbb{P}^1 \times \Delta & \xrightarrow{j} & \text{Univ}_{\widehat{\mathcal{V}}} & \xrightarrow{f} & X \\ p \downarrow & & \downarrow g & & \\ \Delta & \xrightarrow{\iota} & \widehat{\mathcal{V}} & & \end{array}$$

Write $h : \mathbb{P}^1 \times \Delta \rightarrow X$ for the composition $f \circ j$ such that for each $t \in \Delta$, the morphism $h_t : \mathbb{P}^1 = \mathbb{P}^1 \times \{t\} \rightarrow X$ is an embedding as a line in \mathbb{P}^N and the line $h_0(\mathbb{P}^1) \subset X$ is contained in B . Writing $u = h_0(\mathbb{P}^1) \cap f(E)$ and $v = h_0(\mathbb{P}^1) \cap f(F)$, we have $u \neq v$ from $f(E) \cap f(F) = \emptyset$. Since the lines $f(E)$ and $f(F)$ intersect B_o transversally by the requirement $f(z), f(y) \in O$, we have $T_u(B_o) \cap T_u(f(E)) = 0$ and $T_v(B_o) \cap T_v(f(F)) = 0$.

Note that $dh : T(\mathbb{P}^1 \times \Delta) \rightarrow h^*T(X)$ sends the vertical tangent T^p of the projection $p : \mathbb{P}^1 \times \Delta \rightarrow \Delta$ into a line subbundle of $h^*T(X)$, because h_t is an embedding of \mathbb{P}^1 to a line in X for each t . The quotient bundle $\mathcal{N} := h^*T(X)/T^p$ has the property that its restriction to the fiber $h^{-1}(t)$ is isomorphic to the normal bundle $N_{h_t(\mathbb{P}^1) \subset X}$.

The infinitesimal deformation $\frac{\partial}{\partial t}h$ defines a section σ of the vector bundle \mathcal{N} on $\mathbb{P}^1 \times \Delta$. Let k be the vanishing order of σ along $t = 0$, i.e. the nonnegative integer such that $t^{-k}\sigma$ is a holomorphic section of \mathcal{N} which does not vanish identically on $p^{-1}(0)$. Since the complex analytic surface $h(\mathbb{P}^1 \times \Delta)$ contains the germs of the lines $f(E)$ and $f(F)$, the restriction $t^{-k}\sigma|_{h^{-1}(f(E))}$ (resp. $t^{-k}\sigma|_{h^{-1}(f(F))}$) takes values in $T(f(E))$ (resp. $T(f(F))$) modulo T^p . On the other hand, Proposition 8.3 (iv) says that $t^{-k}\sigma|_{t=0}$ must take values in $T(B_o)$ modulo T^p . This implies that the values of $t^{-k}\sigma$ at

$h_0^{-1}(u)$ and at $h_0^{-1}(v)$ must belong to

$$T_u(B_o) \cap T_u(f(E)) = 0 \text{ and } T_v(B_o) \cap T_v(f(F)) = 0.$$

Thus it gives a nonzero section of the normal bundle

$$N_{h_0(\mathbb{P}^1) \subset X} \subset N_{h_0(\mathbb{P}^1) \subset \mathbb{P}^N}$$

of the line $h_0(\mathbb{P}^1) \subset X \subset \mathbb{P}^N$, vanishing at the two distinct points u and v . But the normal bundle of a line in \mathbb{P}^N cannot have a nonzero section vanishing at two distinct points, a contradiction. This proves Theorem 8.5. \square

Proposition 8.6. *In the setting of Theorem 1.6, assume that $\ell' = 1$, then the projection $\Gamma \rightarrow X$ is birational, i.e., the generically finite correspondence Γ defines a rational map $X \dashrightarrow X'$.*

Proof. Recall from Proposition 6.10 that \mathcal{W} and \mathcal{W}' are étale webs of smooth curves. If Γ is not birational to X , then Proposition 7.6 implies that the web \mathcal{W}' is pleated. This is a contradiction to Theorem 8.5. \square

Proposition 8.7. *In the setting of Theorem 1.6, assume that Γ is birational over both X and X' , i.e., it defines a birational map $\Phi : X \dashrightarrow X'$. Then Φ gives a biregular morphism $X \cong X'$.*

Proof. The proof is essentially the same as that of Proposition 4.4 of [HM01], modulo a few minor changes. We reproduce it for the reader's convenience.

Firstly, we claim that there is no hypersurface in X (resp. X') contracted by Φ (resp. Φ^{-1}). Let us prove it for Φ (the same argument works for Φ^{-1}). Assume the contrary and let $H \subset X$ be a hypersurface contracted by Φ , i.e., the proper image $\Phi(H)$ has codimension ≥ 2 in X' . Since X has Picard number 1, all general members of \mathcal{W} intersect H . Since Φ sends germs of members of \mathcal{W} to those of \mathcal{W}' , the proper images under Φ of general members of \mathcal{W} give general members of \mathcal{W}' . It follows that all general members of \mathcal{W}' intersect the variety $f(H)$ of codimension ≥ 2 in X' , a contradiction to the fact that \mathcal{W}' is an étale web of smooth curves.

By the claim, we see that Φ induces a biregular morphism between two quasi-projective varieties $X_o \subset X$ and $X'_o \subset X'$ such that the complement $X \setminus X_o$ and $X' \setminus X'_o$ have codimension ≥ 2 . The isomorphism between the linear systems $H^0(X_o, K_{X_o}^{-k}) \cong H^0(X'_o, K_{X'_o}^{-k})$ induced by Φ for all $k > 0$ extends to an isomorphism $H^0(X, K_X^{-k}) \cong H^0(X', K_{X'}^{-k})$ by Hartogs extension. Since X and X' are Fano by Proposition 6.10, this isomorphism gives a biregular morphism between X and X' . \square

Now we are ready to prove Theorem 1.2 and Theorem 1.8.

Proof of Theorem 1.2. Recall from Proposition 6.10 that lines covering X and X' define étale webs of lines. Applying Theorem 1.6, we have a generically finite correspondence $\Gamma \subset X \times X'$ extending φ . By Proposition 8.6, we know that Γ gives a rational map $X \dashrightarrow X'$. We can apply Proposition 8.6

with X and X' switched to see that Γ gives a birational map $\Phi : X \dashrightarrow X'$. Then Proposition 8.7 implies that Φ gives a biregular morphism. \square

Proof of Theorem 1.8. Let $\Phi : X \rightarrow X'$ be a surjective morphism. Recall from Proposition 6.10 that lines covering X (resp. smooth rational curves of degree ℓ covering X') define an étale web \mathcal{W} (resp. \mathcal{W}') of smooth curves. By Lemma 6.4, the pull-back $\Phi^*\mathcal{W}'$ is an étale web of smooth curves on X . The second assertion of Proposition 6.10 implies that $\Phi^*\mathcal{W}'$ is a subweb of \mathcal{W} . Applying Proposition 8.6 to the webs \mathcal{W}' and $\Phi^*\mathcal{W}'$, with X and X' switched, we see that the graph $\text{Graph}(\Phi) \subset X \times X'$ must be birational to X' . Thus Φ is birational, which must be biregular by Proposition 8.7. \square

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